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On a mixed Littlewood conjecture for quadratic numbers

par Bernard de MATHAN

Résumé. Nous étudions un problème diophantien simultané relié à la conjecture de Littlewood. En utilisant des minorations connues de formes linéaires de logarithmes $p$-adiques, nous montrons qu’un résultat que nous avons précédemment obtenu, concernant les nombres quadratiques, est presque optimal.

Abstract. We study a simultaneous diophantine problem related to Littlewood’s conjecture. Using known estimates for linear forms in $p$-adic logarithms, we prove that a previous result, concerning the particular case of quadratic numbers, is close to be the best possible.

1. Introduction

In a joint paper, with O. Teulié [5], we have considered the following problem. Let $\mathcal{B}=(b_k)_{k\geq 1}$ be a sequence of integers greater than 1. Consider the sequence $(r_n)_{n\geq 0}$, where $r_0=1$ and $r_n = \prod_{0<k \leq n} b_k$ for $n > 0$. For $q \in \mathbb{Z}$, set

$$w_{\mathcal{B}}(q) = \sup\{n \in \mathbb{N} ; q \in r_n\mathbb{Z}\}$$

and

$$|q|_{\mathcal{B}} = \inf\{1/r_n ; q \in r_n\mathbb{Z}\}.$$ 

Notice that $|.|_{\mathcal{B}}$ is not necessarily an absolute value, but when $\mathcal{B}$ is the constant sequence $p$, where $p$ is a prime number, then $|.|_{\mathcal{B}}$ is the usual $p$-adic value.

For $x \in \mathbb{R}$, we denote by $\{x\}$ the number in $[-1/2,1/2[$ such that $x - \{x\} \in \mathbb{Z}$. As usual, we put $\|x\| = |\{x\}|.$

Let $\alpha$ be a real number. Given a positive integer $M$, Dirichlet’s Theorem asserts that for any $n$, there exists an integer $q$, with $0 < q \leq Mr_n$, satisfying simultaneously the approximation condition $\|q\alpha\| < 1/M$ and the divisibility condition $r_n|q$, i. e. $|q|_{\mathcal{B}} \leq 1/r_n$. Indeed, it is enough to
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apply Dirichlet’s Theorem to the number $r_n\alpha$. We thus find positive integers $q$ with

$$q\|q\alpha\| |q|_B < 1.$$ 

By analogy with Littlewood’s conjecture, we ask whether

$$\inf_{q \in \mathbb{N}^*} q\|q\alpha\| |q|_B = 0 \quad (1)$$

holds. The problem is trivial for $\alpha$ rational, and for an irrational number $\alpha$, one can easily see [5] that condition (1) is equivalent to the following: for each $n \in \mathbb{N}$, consider the continued fraction expansion

$$r_n\alpha = [a_{0,n}; a_{1,n}, ..., a_{k,n}...].$$

We have (1) if and only if

$$\sup_{n \geq 0, k \geq 1} a_{k,n} = +\infty.$$ 

However, we shall not use this characterization here.

We do not know whether (1) is satisfied for any real number $\alpha$. In [5], we have proved that if we assume that the sequence $B = (b_k)_{k \geq 1}$ is bounded, (1) is true for every quadratic number $\alpha$. More precisely:

**Theorem 1.1. (de Mathan and Teulié [5])** Suppose that the sequence $B$ is bounded. Let $\alpha$ be a quadratic real number. Then there exists an infinite set of integers $q > 1$ with

$$\|q\alpha\| \ll 1/q \quad (2)$$

and

$$|q|_B \ll 1/\ln q. \quad (3)$$

In particular, we have

$$\liminf_{q \to +\infty} q\ln q\|q\alpha\| |q|_B < +\infty.$$ 

As usual, for positive functions $x$ and $y$, the notation $x \ll y$ means that there exists a positive constant $C$ such that $x \leq Cy$.

In our lecture at Graz, for the “Journées Arithmétiques 2003”, it was discussed whether the factor $\ln q$ in (3) is best possible. We do not know the answer to this question, but we shall prove:
Theorem 1.2. Assume that the sequence $B$ is bounded. Let $\alpha$ be a real quadratic number, and let $S$ be a set of integers $q > 1$ with

$$\|qa\| \ll 1/q.$$  \tag{2}

Then there exists a constant $\lambda = \lambda(S)$ such that

$$|q|B \gg \frac{1}{(\ln q)^\lambda}$$  \tag{4}

for any $q \in S$.

One may expect that (4) holds for any $\lambda > 1$, but we are not able to prove this. We do not even know whether there exists a real number $\lambda$ for which (4) holds for any set $S$ of integers $q > 1$ satisfying (2). Indeed, Theorem 1.2 does not ensure that $\sup_S \lambda(S) < +\infty$.

There is some analogy between this problem, and the classical simultaneous Diophantine approximation. For instance, let us recall Peck’s Theorem. Let $n$ be an integer greater than 1, and let $\alpha_1, ..., \alpha_n$, be $n$ numbers in a real algebraic number field of degree $n + 1$ over $\mathbb{Q}$. Then it was proved by Peck [7] that there exists an infinite set of integers $q > 1$ with

$$\|qa_k\| \ll (\ln q)^{-1/(n-1)} q^{-1/n}$$

for $1 \leq k < n$, and

$$\|qa_n\| \ll q^{-1/n}.$$  \tag{5}

Assume that $1, \alpha_1, ..., \alpha_n$ are linearly independent over $\mathbb{Q}$, and let $S$ be an infinite set of integers $q > 1$, with

$$\|qa_k\| \ll q^{-1/n}$$

for each $1 \leq k \leq n$. Then we have proved in [3] that there exists a constant $\kappa = \kappa(S)$ such that

$$\max_{1 \leq k < n} \|qa_k\| \gg (\ln q)^{-\kappa} q^{-1/n}.$$  \tag{6}

Theorem 1.2 can be regarded as an analogue of this result with $n = 1$, and its proof is similar.

2. Proof of the result

2.1. Some rational approximations of $\alpha$.

In the quadratic field $\mathbb{Q}(\alpha)$, there exists a unit $\omega$ of infinite order. Replacing, if necessary, $\omega$ by $\omega^2$ or $1/\omega^2$, we may suppose $\omega > 1$. In his original work, Peck uses units which are “large” and whose other conjugates are “small” and close to be equal. Here, Peck’s units are just the $\omega_m$’s, with $m \in \mathbb{N}$. We shall use these units in order to describe the rational approximations of $\alpha$ which satisfy (2).
Denote by $\sigma_0 = \text{id}$ and $\sigma_1 = \sigma$ the automorphisms of $\mathbb{Q}(\alpha)$. As usual, we denote by $\text{Tr}$ the trace form $\text{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}} = \sigma_0 + \sigma_1$. The basis $(1, \alpha)$ of $\mathbb{Q}(\alpha)$ admits a dual basis $(\beta_0, \beta_1)$ for the non-degenerate $\mathbb{Q}$-bilinear form $(x, y) \mapsto \text{Tr}(xy)$ on $\mathbb{Q}(\alpha)$. That means that, if we set $\alpha_0 = 1$ and $\alpha_1 = \alpha$, we have $\text{Tr}(\alpha_k\beta_l) = \delta_{kl}$, for $k = 0, 1$ and $l = 0, 1$, where $\delta_{ll} = 1$, and $\delta_{kl} = 0$ if $k \neq l$. Here it is easy to calculate $\eta = \frac{-q\sigma(\alpha) + q'}{\alpha - \sigma(\alpha)}$, where $q$ and $q'$ are rational numbers, we have

$$q = \text{Tr}\eta$$  \hspace{1cm} (5)

and

$$q' = \text{Tr}(\alpha\eta).$$  \hspace{1cm} (6)

Also notice that (5) and (6) imply that

$$qa - q' = (\alpha - \sigma(\alpha))\sigma(\eta).$$  \hspace{1cm} (7)

Let $D$ be a positive integer such that $D\alpha$, $D\alpha\sigma(\alpha)$, and $D\alpha\alpha\sigma(\alpha)$ are algebraic integers.

The notation $A \asymp B$, where $A$ and $B$ are positive quantities, means that $B \ll A \ll B$.

**Lemma 2.1.** Let $\gamma$ be a positive number in $\mathbb{Q}(\alpha)$. Let $\Delta$ be a positive integer such that $\Delta\gamma$ is an algebraic integer. For each $m \in \mathbb{N}$, define the rational number

$$q = q(m) = \text{Tr}(\gamma\omega^m).$$  \hspace{1cm} (8)

Then $\Delta q$ is a rational integer, one has $q > 0$ when $m$ is large, and the integers $D\Delta q$ satisfy (2).

**Proof.** Also define

$$q' = q'(m) = \text{Tr}(\alpha\gamma\omega^m).$$

As $\Delta\gamma\omega^m$ and $D\Delta\alpha\gamma\omega^m$ are algebraic integers, $\Delta q$ and $D\Delta q'$ are rational integers. As $\sigma(\omega) = 1/\omega$, we have $q = \gamma\omega^m + \sigma(\gamma)\omega^{-m}$, hence $q > 0$ as soon as $\omega^{2m} > -\sigma(\gamma)/\gamma$, and then

$$q \asymp \omega^m.$$  \hspace{1cm} (9)

From (7), we get $qa - q' = (\alpha - \sigma(\alpha))\sigma(\gamma)\omega^{-m}$, hence

$$|qa - q'| \asymp \omega^{-m}.$$  \hspace{1cm} (10)

As $D\Delta q$ and $D\Delta q'$ are integers, it follows from (10) that for large $m$ we have $\|D\Delta qa\| = D\Delta|qa - q'|$, and by (9) and (10), the integers $D\Delta q$ satisfy (2).
Conversely:

**Lemma 2.2.** Let $S$ be a set of positive integers $q$ satisfying (2). Then there exists a finite set $\Gamma$ of numbers $\gamma \in \mathbb{Q}(\alpha)$, $\gamma \neq 0$, such that for any $q \in S$, there exist $\gamma \in \Gamma$ and $m \in \mathbb{N}$ such that

$$q = \text{Tr}(\gamma \omega^m).$$

**Proof.** For $q \in S$, let $m(q) = m$ be the positive integer such that $\omega^{m-1} \leq q < \omega^m$. We thus have $\omega^m \asymp q$. Let $q'$ be the rational integer such that $\{q\alpha\} = q\alpha - q'$. Set

$$\gamma = \frac{-q\sigma(\alpha) + q'}{\alpha - \sigma(\alpha)} \omega^{-m}.$$

First, notice that $D\gamma$ is an algebraic integer. From (5), we get (8). Writing

$$\gamma \omega^m = q - \frac{q\alpha - q'}{\alpha - \sigma(\alpha)}$$

we see that $\gamma > 0$ when $q$ is large, and $\gamma \omega^m \asymp q$. As we have $\omega^m \asymp q$, we thus get $\gamma \asymp 1$. We also have

$$\sigma(\gamma) = q\frac{q\alpha - q'}{\alpha - \sigma(\alpha)} \omega^m,$$

hence, by (2), $|\sigma(\gamma)| \ll \omega^m / q$, and thus, $|\sigma(\gamma)| \ll 1$. Then, as $D\gamma$ is an algebraic integer in $\mathbb{Q}(\alpha)$, and $\max(|\gamma|, |\sigma(\gamma)|) \ll 1$, the set of the $\gamma$'s is finite.

2.2. End of proof.

Denote by $P$ the set of all prime numbers dividing one of the $b_k$. Since we assume that the sequence $(b_k)$ is bounded, this set is finite. For $p \in P$, we extend the $p$-adic absolute value to $\mathbb{Q}(\alpha)$. The completion of this field is $\mathbb{Q}_p(\alpha)$. As above, let $\omega$ be a unit in $\mathbb{Q}(\alpha)$ with $\omega > 1$. Note that $|\omega|_p = 1$. The ball $\{x \in \mathbb{Q}_p(\alpha); |x - 1|_p < p^{-1/(p-1)}\}$ is a subgroup of finite index in the multiplicative group $\{x \in \mathbb{Q}_p(\alpha); |x|_p = 1\}$. Hence, replacing $\omega$ by $\omega^n$, where $n$ is a suitable positive integer, we may also suppose that $|\omega - 1|_p < p^{-1/(p-1)}$ for every $p \in P$.

We shall use the $p$-adic logarithm function, which is defined on the multiplicative group $\{x \in \mathbb{C}_p; |x - 1|_p < 1\} \subset \mathbb{C}_p$ by

$$\log x = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{(x - 1)^n}{n}.$$

This function satisfies

$$\log xy = \log x + \log y,$$
and, for $|x - 1|_p < p^{-1/(p-1)}$, $|\log x|_p = |x - 1|_p$. Hence, for $|x - 1|_p < p^{-1/(p-1)}$ and $|y - 1|_p < p^{-1/(p-1)}$, we have

$$|\log x - \log y|_p = |\log \frac{x}{y}|_p = |\frac{x}{y} - 1|_p = |x - y|_p. \quad (11)$$

We prove:

**Lemma 2.3.** Let $p$ be a number of $P$. Let $\gamma$ be a positive number of $\mathbb{Q}(\alpha)$. For $m \in \mathbb{N}$, set

$$q = q(m) = \text{Tr}(\gamma\omega^m). \quad (8)$$

Then, if

$$|\frac{\sigma(\gamma)}{\gamma} + 1|_p \geq p^{-1/(p-1)},$$

we have

$$|q|_p \asymp 1$$

for large $m$; if

$$|\frac{\sigma(\gamma)}{\gamma} + 1|_p < p^{-1/(p-1)},$$

then

$$|q|_p \asymp |2m \log \omega - \log(-\sigma(\gamma)/\gamma)|_p. \quad (12)$$

**Proof.** Recall that $q > 0$ when $m$ is large (Lemma 2.1). From the definition, we get for each $p \in P$, $|q|_p = |\gamma\omega^m + \sigma(\gamma)\omega^{-m}|_p = |\gamma|_p|\omega^{2m} - \delta|_p$, where $\delta = -\sigma(\gamma)/\gamma$. If $|\delta - 1|_p \geq p^{-1/(p-1)}$, we have $|\omega^{2m} - \delta|_p \geq p^{-1/(p-1)}$, since $|\omega - 1|_p < p^{-1/(p-1)}$ and $|\omega^{2m} - 1|_p < p^{-1/(p-1)}$. Then we get

$$|q|_p \asymp 1.$$ 

If $|\delta - 1|_p < p^{-1/(p-1)}$, then, by (11), we write $|\omega^{2m} - \delta|_p = |2m \log \omega - \log \delta|_p$, and we obtain (12).

Accordingly, in order to achieve the proof of the result, we shall use known lower bounds for linear forms in $p$-adic logarithms. For instance, it follows from [8] that:

**Lemma 2.4.** (K. Yu [8]) Let $x$ and $y$ be algebraic numbers in $\mathbb{C}_p$, with $|x - 1|_p < p^{-1/(p-1)}$ and $|y - 1|_p < p^{-1/(p-1)}$. Then there exists a real constant $\kappa$ such that for any pair $(k, \ell)$ of rational integers with $k \log x + \ell \log y \neq 0$, one has

$$|k \log x + \ell \log y|_p \gg (\max(|k|, |\ell|))^{-\kappa}.$$

Note that this result is trivial, with $\kappa = 1$, if $\log x$ and $\log y$ are not linearly independent over $\mathbb{Q}$, and $\log x \neq 0$, i.e., $x \neq 1$. Indeed, if $a \log x = b \log y$, where $a$ and $b$ are rational integers with $b \neq 0$, then we write $|k \log x + \ell \log y|_p = \frac{1}{|b|_p}|bk + a\ell|_p |x - 1|_p$. Hence we get $|k \log x + \ell \log y|_p \gg |bk + a\ell|_p \geq |bk + a\ell|^{-1} \gg (\max(|k|, |\ell|))^{-1}$, when $k \log x + \ell \log y \neq 0$. 

We can then achieve the proof of Theorem 1.2. Applying Lemma 2.2, we can suppose that the set $\Gamma$ contains a unique element $\gamma > 0$, i.e., for any $q \in \mathcal{S}$, there exists $m \in \mathbb{N}$ such that we have (8). It follows from Lemma 2.3 and 2.4 that there exists a constant $\kappa$ such that $|q|_p \geq m^{-\kappa}$ (one may take $\kappa = 0$ if $|\sigma(\gamma) + 1|_p \geq p^{-1/(p-1)}$). As $q \asymp \omega^m$, hence $m \asymp \ln q$, we get $|q|_p \geq (\ln q)^{-\kappa}$. Now set $\kappa = \kappa_p$ (the constant $\kappa_p$ may depend upon $p \in \mathcal{P}$). Note that $|q|_B \geq \prod_{p \in \mathcal{P}} |q|_p$. Indeed, putting $|q|_B = 1/r_n$, we have $q \in r_n \mathbb{Z}$, hence $|q|_p \leq |r_n|_p$ and $\prod_{p \in \mathcal{P}} |q|_p \leq \prod_{p \in \mathcal{P}} |r_n|_p = 1/r_n$. We thus get (4) with $\lambda = \sum_{p \in \mathcal{P}} \kappa_p$, and Theorem 1.2 is proved.

2.3. A remark.

Note that one may also use Lemma 2.3 for solving the opposite problem. For simplicity, consider the case where $|.|_B$ is the $p$-adic value for a prime number $p$. If we take a positive number $\gamma \in \mathbb{Q}(\alpha)$ such that $\sigma(\gamma) = -\gamma$, for instance, $\gamma = \alpha - \sigma(\alpha)$ (one may replace $\alpha$ by $-\alpha$, and so, we can suppose $\alpha - \sigma(\alpha) > 0$), then we have $\log(-\sigma(\gamma)/\gamma) = 0$, and by (12), we get $|\text{Tr}(\gamma \omega^m)|_p \ll |m|_p$. By Lemma 2.1, there exists a positive integer $\lambda$ such that for every large $m$, the numbers $q = q(m) = A\text{Tr}(\gamma \omega^m)$ are positive integers satisfying (2). For $m = p^s$ with $s \in \mathbb{N}$, we get $|m|_p = 1/m$, hence $|q|_p \asymp 1/m$. Since $m \asymp \ln q$, we have thus proved that there exists an infinite set of integers $q > 1$ satisfying (2) and (3) (which is Theorem 1.1). In this way we obtain integers $q > 1$ satisfying (2) and such that $|q|_p \asymp 1/\ln q$.

One can ask whether there exists an infinite set of integers $q > 1$ satisfying (2), with

$$\inf |q|_p \ln q = 0. \quad (3')$$

Given a positive decreasing sequence $(\epsilon_m)$ with $\sum_{m=0}^{+\infty} \epsilon_m = +\infty$, a $p$-adic version [4] of Khintchine’s Theorem ensures that for almost all $x \in \mathbb{Z}_p$, there exist infinitely many positive integers $m$ such that $|x - m|_p \leq \epsilon_m$. One often considers as reasonable the hypothesis that a given “special” irrational number $x \in \mathbb{Z}_p$ satisfies this condition, with $\epsilon_m = 1/(m \ln m)$ for $m > 1$ (which is false if $x \in \mathbb{Z}_p \cap \mathbb{Q}$, since in this case, we have $|x - m|_p \gg 1/m$ for $m$ large). Let us prove that we can choose $\gamma > 0$ in $\mathbb{Q}(\alpha)$, with $|\sigma(\gamma) + 1|_p < |\omega - 1|_p$, such that $\frac{\log(-\sigma(\gamma)/\gamma)}{\log \omega}$ is an irrational number in $\mathbb{Z}_p$. In order to make this obvious, we prove:

**Lemma 2.5.** There exists $\xi \in \mathbb{Q}(\alpha)$ such that $\xi$ is not a unit, $N_{\mathbb{Q}(\alpha):\mathbb{Q}} \xi = 1$, and $|\xi|_p = 1$.

**Proof.** The number $\omega$ is a root of the equation $\omega^2 - S\omega + 1 = 0$, where $S$ is a rational integer, $S = \text{Tr} \omega$. The number $\xi$ must be a root of an equation $\xi^2 - t\xi + 1 = 0$, where $t$ is a rational number for which there exists a positive
rational number $\rho$ such that $t^2 - 4 = \rho^2(S^2 - 4)$. Such pairs $(t, \rho)$ can be expressed by using a rational parameter $\theta$:

$$
t = \frac{2(S^2 - 4)\theta^2 + 2}{(S^2 - 4)\theta^2 - 1} = 2 + \frac{4}{(S^2 - 4)\theta^2 - 1}
$$

$$
\rho = \frac{4\theta}{(S^2 - 4)\theta^2 - 1}.
$$

Let us show that we can choose $\theta \in \mathbb{Q}^*$ such that $t \not\in \mathbb{Z}$ and $|t|_p \leq 1$. It is enough to take $\theta = p$. As we have $S^2 > 4$, hence $S^2 \geq 9$ and $(S^2 - 4)p^2 - 1 > 4$, $t$ cannot be an integer for this choice of $\theta$. We have $|t|_p \leq 1$, since $|(S^2 - 4)p^2 - 1|_p = 1$. Then there exists a number $\xi \in \mathbb{Q}(\alpha)$ such that $\xi^2 - t\xi + 1 = 0$, and $\xi$ is neither a rational number, since $\rho > 0$, nor an algebraic integer, since $t \not\in \mathbb{Z}$. Then we have $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\xi) = 1$, and $|\xi|_p = 1$ because either condition $|\xi|_p < 1$ or $|\xi|_p > 1$ would imply $|t|_p = |\xi + \xi^{-1}|_p > 1$.

Replacing $\xi$ by $\xi^n$, where $n$ is a suitable positive integer, we thus may find a $\xi$ satisfying Lemma 2.5, with moreover $|\xi - 1|_p < |\omega - 1|_p$. Then we have $|\log \xi|_p < |\log \omega|_p$. Further let us prove that $\frac{\log \xi}{\log \omega} \in \mathbb{Q}_p$. Indeed that is trivial if $\alpha \in \mathbb{Q}_p$, since in this case $\xi$ and $\omega$ lie in $\mathbb{Q}_p$, hence so do $\log \xi$ and $\log \omega$. If $\mathbb{Q}_p(\alpha)$ has degree 2 over $\mathbb{Q}_p$, then $\log \xi$ and $\log \omega$ lie in $\mathbb{Q}_p(\alpha)$. But $\sigma$ can be extended into a continuous $\mathbb{Q}_p$-automorphism of $\mathbb{Q}_p(\alpha)$, and we get $\sigma(\frac{\log \xi}{\log \omega}) = \frac{\log \sigma(\xi)}{\log \sigma(\omega)} = \frac{-\log \xi}{-\log \omega} = \frac{\log \xi}{\log \omega}$, since $\xi \sigma(\xi) = \omega \sigma(\omega) = 1$. That proves that $\frac{\log \xi}{\log \omega} \in \mathbb{Q}_p$, and since $|\log \xi|_p < |\log \omega|_p$, we conclude that $\frac{\log \xi}{\log \omega} \in \mathbb{Z}_p$. Lastly, $\frac{\log \xi}{\log \omega}$ is not a rational number, since $\xi$ is not a unit. Now, by Hilbert’s Theorem, there exists $\gamma \in \mathbb{Q}(\alpha)$, with $\gamma > 0$, such that $\xi = -\sigma(\gamma)/\gamma$. We thus have found $\gamma > 0$ in $\mathbb{Q}(\alpha)$, such that $|\frac{\sigma(\gamma)}{\gamma} + 1|_p < p^{-1/(p-1)}$ and $\frac{\log(-\sigma(\gamma)/\gamma)}{2\log \omega}$ is an irrational element of $\mathbb{Z}_p$. Under the above hypothesis, it would exist infinitely many integers $m > 1$ with $|\frac{\log(-\sigma(\gamma)/\gamma)}{2\log \omega} - m|_p \ll 1/(m \log m)$, and, by (12), we could obtain an infinite set of integers $q > 1$, $q = A\text{Tr}(\gamma \omega^m)$ where $A$ is a positive integer, satisfying (2) and such that $|q|_p \ll \frac{1}{\ln q \ln \ln q}$. In particular, (3') would be satisfied.

3. Conclusion

For a sequence $\mathcal{B}$ bounded, the Roth-Ridout Theorem [6] allows us to see that for any irrational algebraic real number $\alpha$, thus in particular for $\alpha$ quadratic, we have:

$$
\inf_{q > 0} q^{1+\epsilon} \|q\alpha\||q|_B > 0
$$
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(see [5]). Of course, our method is far from enabling us to prove that there exists a real constant $\lambda$ such that

$$\inf_{q > 1} q (\ln q)^\lambda \|q\alpha\| |q|_B > 0.$$ 

We can only study the approximations with $q\|q\alpha\| \ll 1$. It seems difficult to study approximations in the “orthogonal direction” $q|q|_B \ll 1$, with for instance, $q = p^n$, for a prime number $p$. For such approximations, it is not known whether $\inf_{n \in \mathbb{N}} \|p^n \alpha\| = 0$ holds, neither if there exists $\lambda$ such that $\inf_{n > 0} n^\lambda \|p^n \alpha\| > 0$. It is very difficult to obtain more precise results than the Roth-Ridout Theorem (see [1]).

Even for rational approximations satisfying (2), we are not able to prove that the constants $\lambda(S)$ are bounded. This is related to Lemma 2.4. It would be necessary to prove that there exists a real constant $\kappa$ for which this Lemma holds for $x = \omega$ and for any $y \in \mathbb{Q}(\alpha)$ with $|y - 1|_p < p^{-1/(p-1)}$ and $N_{\mathbb{Q}(\alpha)/\mathbb{Q}}(y) = 1$. There exist many effective estimates of $|k \log x + \ell \log y|_p$ (see for instance [2] and [8]), but they do not provide the needed result. It seems difficult to take the particular conditions required into account.

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References


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