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<http://jtnb.cedram.org/item?id=JTNB_2007__19_3_567_0>
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Abstract. In recent years, starting with the paper [B-D-S], we have investigated the possibility of characterizing countable subgroups of the torus $T = \mathbb{R}/\mathbb{Z}$ by subsets of $\mathbb{Z}$. Here we consider new types of subgroups: let $K \subseteq T$ be a Kronecker set (a compact set on which every continuous function $f : K \to T$ can be uniformly approximated by characters of $T$), and $G$ the group generated by $K$. We prove (Theorem 1) that $G$ can be characterized by a subset of $\mathbb{Z}^2$ (instead of a subset of $\mathbb{Z}$). If $K$ is finite, Theorem 1 implies our earlier result in [B-S]. We also prove (Theorem 2) that if $K$ is uncountable, then $G$ cannot be characterized by a subset of $\mathbb{Z}$ (or an integer sequence) in the sense of [B-D-S].

1. Introduction

Let $T = \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ denotes the additive group of the real numbers, $\mathbb{Z}$ is its subgroup consisting of the integers. If $x \in \mathbb{R}$, then $\|x\|$ denotes its distance to the nearest integer; this function is constant on cosets by $\mathbb{Z}$, so it is well-defined on $T$. A set $K \subseteq T$ is called a Kronecker set if it

Manuscrit reçu le 13 mai 2005.
Research partially supported by the Hungarian National Foundation for Scientific Research (OTKA) Grants No. T032236, T 042750, T043623 and T049693.
is nonempty, compact, and for every continuous function $f : K \to T$ and $\delta > 0$ there is an $n \in \mathbb{Z}$ such that

$$\max_{\alpha \in K} \| f(\alpha) - n\alpha \| < \delta.$$  

If $K \subseteq T$ is a finite set, it is a Kronecker set if and only if its elements are independent over $\mathbb{Z}$ (this is essentially Kronecker’s classical theorem on simultaneous diophantine approximation). There are many uncountable Kronecker sets, see e.g. [L-P], Ch. 1.

In [B-D-S] and in [B-S], we proved for a subgroup $G \subseteq T$ generated by a finite Kronecker set that $G$ can be characterized by a subset of the integers in certain ways. In fact we dealt with any countable subgroup of $T$ in [B-D-S], and the result of [B-S] was generalized also for any countable subgroup in [B]. For further generalizations and strengthenings of these results, see [Bi1], [Bi2], [D-M-T], [D-K], [B-S-W].

In the present paper, we prove such a characterization of a group generated by a general Kronecker set by a subset of $\mathbb{Z}^2$ (instead of a subset of $\mathbb{Z}$). We also show, on the contrary, that using a subset of $\mathbb{Z}$, the characterization is impossible, if $K$ is uncountable. More precisely, we prove the following results.

Throughout the paper, let $K$ be a fixed Kronecker set, $G$ the subgroup of $T$ generated by $K$, and let $\epsilon > 0$ be a fixed number. Write

$$l(x) = \frac{-1}{\log_2 x} \quad \text{for } 0 < x < 1/2,$$

and extend it to every $x \geq 0$ by $l(0) = 0$, and $l(x) = 1$ for $x \geq 1/2$.

**Theorem 1.** There is an infinite subset $A \subseteq \mathbb{Z}^2$ such that for every $\alpha \in G$ we have

$$\sum_{n=(n_1,n_2) \in A} l^{1+\epsilon} \left( \min \left( \|n_1\alpha\|,\|n_2\alpha\| \right) \right) < \infty, \quad (1.1)$$

and if $\beta \in T$ satisfies

$$\min \left( \|n_1\beta\|,\|n_2\beta\| \right) < \frac{1}{10} \quad (1.2)$$

for all but finitely many $n = (n_1,n_2) \in A$, then $\beta \in G$. Moreover, $A$ has the additional property that if $\alpha_1,\alpha_2,\ldots,\alpha_t \in G$ are finitely many given elements, then there is a function $f : A \to \mathbb{Z}$ such that $f(n) = n_1$ or $f(n) = n_2$ for every $n = (n_1,n_2) \in A$, and for every $1 \leq i \leq t$ we have

$$\sum_{n \in A} l^{1+\epsilon} \left( \|f(n)\alpha_i\| \right) < \infty. \quad (1.3)$$

If $K$ is finite, the theorem of [B-S] follows at once from Theorem 1, since we can take all elements of $K$ as $\alpha_1,\alpha_2,\ldots,\alpha_t$ (see also Lemma 2 (i) in Section 3). Note that the statement of the Theorem in [B-S] contains a misprint: $\lim \inf$ should be replaced by $\lim \sup$ there.
Theorem 2. If $K$ is uncountable, and $A \subseteq \mathbb{Z}$ is an infinite subset, then

$$G \neq \left\{ \beta \in T : \lim_{n \in A} \| n\beta \| = 0 \right\}.$$ 

This is in fact an easy corollary of a result of Aaronson and Nadkarni, but since the proof of that result is very sketchy in [A-N], we present its proof (see Section 4, Prop. 1.).

We give the proof of Theorem 1 in Section 2. We mention that the basic idea is the same as in [Bi2]. Some lemmas needed in the proof of Theorem 1 are presented in Section 3. We remark that Lemma 4 is very important in the proof, and it provides the main reason why we need an $\epsilon > 0$ in the theorem. The proof of Theorem 2 is given in Section 4. Section 5 contains a few comments and open questions.

2. Proof of Theorem 1

We will use Lemmas 2, 3 and 4, these lemmas are stated and proved in Section 3, so see that section if we refer to one of these lemmas.

If $x \in \mathbb{R}$, we also write $x$ for the coset of $x$ modulo $\mathbb{Z}$, so we consider $x$ as an element of $T$. The fractional part function $\{x\}$ is well-defined on $T$.

Let $T^{(2)}$ be the subgroup of $T$ defined by

$$T^{(2)} = \left\{ \frac{a}{2^N} : N \geq 0, 1 \leq a \leq 2^N \right\}.$$ 

For $N \geq 0$ and $1 \leq a \leq 2^N$ let

$$K_{N,a} = \left\{ \alpha \in K : \frac{a-1}{2^N} < \{\alpha\} < \frac{a}{2^N} \right\}.$$ 

Since $K$ is a Kronecker set, we can easily see that $K \cap T^{(2)} = \emptyset$, and so every $K_{N,a}$ is an open-closed subset of $K$, and

$$K = \bigcup_{a=1}^{2^N} K_{N,a}$$

(disjoint union). Let $F$ be the set of functions $f : K \to T^{(2)}$ which are constant on each small set of one of these subdivisions, i.e.

$$F = \left\{ f : K \to T^{(2)} : |f(K_{N,a})| \leq 1 \text{ for some } N \geq 0 \text{ and for every } 1 \leq a \leq 2^N \right\},$$

where $|f(K_{N,a})|$ denotes the cardinality of the set $f(K_{N,a})$, and we write $\leq 1$ because it may happen that some set $K_{N,a}$ is empty. Observe that $F$ is countable. Every element of $F$ is a continuous function on $K$, and $F$ is a group under pointwise addition. For a pair $(N,a)$ with $N \geq 0$ and $1 \leq a \leq 2^N$ let $F_{N,a} \leq F$ be the subgroup

$$F_{N,a} = \{ f \in F : f(\alpha) = 0 \text{ for } \alpha \in K \setminus K_{N,a}, |f(K_{N,a})| \leq 1 \}.$$
For any \( N \geq 0 \) let \( g_N \in F \) be defined by
\[
g_N(\alpha) = \frac{a}{2^N} \text{ for every } \alpha \in K_{N,a} \text{ and for every } 1 \leq a \leq 2^N,
\]
and let \( f_{N,a,r} \in F_{N,a} \) be defined by \((N \geq 0, 1 \leq a \leq 2^N, r \geq 1 \) are fixed):
\[
f_{N,a,r}(\alpha) = \begin{cases} 2^{-r}, & \text{if } \alpha \in K_{N,a} \\ 0, & \text{if } \alpha \in K \setminus K_{N,a}. \end{cases}
\]
Clearly
\[
\max_{\alpha \in K} \|g_N(\alpha) - \alpha\| \leq 2^{-N} \text{ for every } N \geq 0. \tag{2.1}
\]
Remark that the functions \( g_N \) are not necessarily distinct, but if \( N \geq 0 \) is fixed, then
\[
|\{\nu \geq 0 : g_\nu = g_N\}| < \infty, \tag{2.2}
\]
since otherwise (2.1), applied for the elements \( \nu \) of this set, would give \( g_N(\alpha) = \alpha \) for every \( \alpha \in K \), which is impossible by \( K \cap T^{(2)} = \emptyset \).

For every \( f \in F \) take a number \( C(f) > 0 \), and for every \( N \geq 0 \) a number \( R(N) > 0 \), we assume the following inequalities:
\[
\sum_{f \in F} C(f)^{-\epsilon} < \infty, \quad \sum_{N=0}^{\infty} R(N)^{-\epsilon} < \infty, \tag{2.3}
\]
and (it is possible by (2.2)):
\[
C(g_N) > N \text{ for every } N \geq 0. \tag{2.4}
\]
For every \( f \in F \) and for every integer \( j \geq 1 \) we take an integer \( m_j(f) \) such that
\[
\max_{\alpha \in K} \|f(\alpha) - m_j(f)\alpha\| < 2^{-j-2^{j}C(f)}, \tag{2.5}
\]
which is possible, since \( K \) is a Kronecker set. Moreover, we can assume that if \( j, j^* \geq 1, f, f^* \in F \), then
\[
m_j^*(f^*) \neq m_j(f) \text{ if } (j, f) \neq (j^*, f^*). \tag{2.6}
\]
Indeed, there are countably many pairs \((j, f)\), and for a fixed pair \((j, f)\) there are infinitely many possibilities for \( m_j(f) \) in (2.5), so we can define recursively the integers \( m_j(f) \) to satisfy (2.5) and (2.6).

Let \( j(N,a,r) \geq 1 \) be integers for every triple \( (N,a,r) \in V \), where
\[
V = \left\{ (N,a,r) : N \geq 0, 1 \leq a \leq 2^N, r > R(N) \right\},
\]
satisfying that if \((N^*,a^*,r^*) \in V \) is another such triple, then
\[
j(N,a,r) \neq j(N^*,a^*,r^*), \text{ if } (N,a,r) \neq (N^*,a^*,r^*). \tag{2.7}
\]
We easily see from (2.6) and (2.7) that for \((N,a,r),(N^*,a^*,r^*) \in V\) we have
\[ m_j(N,a,r)(f_{N,a,r}) \neq m_j(N^*,a^*,r^*)(f_{N^*,a^*,r^*}), \text{ if } (N,a,r) \neq (N^*,a^*,r^*). \] (2.8)

Define
\[ H_1 = \left\{ m_j(N,a,r)(f_{N,a,r}) : (N,a,r) \in V \right\}. \] (2.9)

We claim that
\[ \sum_{n \in H_1} l^{1+\epsilon}(\|n\alpha\|) < \infty \] (2.10)

for every \(\alpha \in K\). Indeed, let \(\alpha \in K\) be fixed. We have
\[ \left\| m_j(N,a,r)(f_{N,a,r})\alpha \right\| \leq \|f_{N,a,r}(\alpha)\| + 2^{-1-2^j(N,a,r)C(f_{N,a,r})} \] (2.11)

by (2.5). Now, on the one hand,
\[ \sum_{a=1}^{2^N} l^{1+\epsilon}(\|f_{N,a,r}(\alpha)\|) = l^{1+\epsilon}(2^{-r}) \sum_{N=0}^{\infty} \sum_{r>R(N)} l^{1+\epsilon}(2^{-r}) < \infty \] (2.12)

by (2.3); on the other hand, using (2.7) and (2.3), we get
\[ \sum_{(N,a,r) \in V} l^{1+\epsilon}(2^{-1-2^j(N,a,r)C(f_{N,a,r})}) \leq \sum_{f \in F} \sum_{j \geq 1} (C(f)2^j)^{-(1+\epsilon)} < \infty. \] (2.13)

In view of Lemma 2 (i), (2.11)-(2.13), and the definition of \(H_1\) in (2.9), we get (2.10).

If \(s\) is a nonnegative integer, the following set is a compact subset of \(T\):
\[ K_s = \left\{ \alpha = \sum_{i=1}^{t} k_i\alpha_i : t \geq 1, \alpha_1,\alpha_2,\ldots,\alpha_t \in K, \right\} \]

\[ k_1, k_2, \ldots, k_t \in \mathbb{Z}, \sum_{i=1}^{t} |k_i| \leq s \}

Lemma 1. There is a subset \(H\) of the integers such that \(H_1 \subseteq H\) and on the one hand we have
\[ \sum_{n \in H} l^{1+\epsilon}(\|n\alpha\|) < \infty \] (2.14)

for every \(\alpha \in K\); on the other hand, if \(\beta \in T\) has the property that
\[ \|n\beta\| < \frac{1}{10} \] (2.15)

for all but finitely many \(n \in H\), then there is a group homomorphism \(\phi : F \to T\) which satisfies the following properties:

(i) for all but finitely many pairs \((f,j)\) with \(f \in F, j \geq 1\) we have
\[ \|\phi(f) - m_j(f)\beta\| < 2^{-C(f)-j}; \] (2.16)
(ii) for every \((N, a)\) pair with \(N \geq 0, 1 \leq a \leq 2^N\), if \(K_{N,a} \neq \emptyset\), there is a unique integer \(k_{N,a}\) for which
\[
\phi(f) = k_{N,a}f(\alpha)
\] (2.17)
for every \(f \in F_{N,a}\), where \(\alpha \in K_{N,a}\) is arbitrary; if \(K_{N,a} = \emptyset\), we put \(k_{N,a} = 0\), and then for large \(N\) we have
\[
\max_{1 \leq a \leq 2^N} |k_{N,a}| \leq 2^{R(N)};
\] (2.18)

(iii) if \(N\) is large enough, then writing \(s = \sum_{a=1}^{2^N} |k_{N,a}|\), there is an \(\alpha \in K_s\) such that
\[
\|\alpha - \beta\| \leq \frac{1}{N} + s2^{-N}.
\] (2.19)

Proof. Define
\[
H_2 = \{2^r (m_{j+1}(f) - m_j(f)) : f \in F, j \geq 1, 0 \leq r \leq j - 1 + C(f)\}.
\]
Let us choose for every triple \(f_1, f_2, f_3 \in F\) with \(f_3 = f_1 + f_2\) an infinite subset \(J_{f_1,f_2,f_3}\) of the positive integers such that (the first summation below is over every such triple from \(F\))
\[
\Sigma := \sum_{f_3=f_1+f_2} \sum_{j \in J_{f_1,f_2,f_3}} \left(2^j \min(C(f_1), C(f_2), C(f_3))\right)^{-\epsilon} < \infty.
\] (2.20)
Since \(C(f) > 0\) for every \(f \in F, \epsilon > 0\) and \(F\) is countable, this is obviously possible. Then define (we mean again that \(f_1, f_2, f_3\) run over every such triple from \(F\))
\[
H_3 = \left\{2^r (m_j(f_1) + m_j(f_2) - m_j(f_3)) : f_3 = f_1 + f_2, j \in J_{f_1,f_2,f_3}, \ 0 \leq r \leq j - 2\right\},
\]
\[
H_4 = \{2^r (m_1(g_N) - 1) : N \geq 1, 0 \leq r \leq \log_2 N\}.
\]
Let \(H = \bigcup_{i=1}^4 H_i\). We first prove (2.14). If \(f \in F, j \geq 1\) and \(\alpha \in K\), then
\[
\|\langle m_{j+1}(f) - m_j(f)\rangle\alpha\| \leq 2^{-j+C(f)-1-2(j-1)\epsilon C(f)}
\] (2.21)
by (2.5), therefore, using also Lemma 2 (ii) and (2.3), we obtain
\[
\sum_{n \in H_2} \max_{\alpha \in K} l^{1+\epsilon} (\|n\alpha\|) \leq m \sum_{f \in F} \sum_{j \geq 1} C(f)^{-\epsilon (2^j - 1) - \epsilon} < \infty.
\] (2.22)
If \(\alpha \in K, f_1, f_2, f_3 \in F, f_3 = f_1 + f_2\) and \(j \in J_{f_1,f_2,f_3}\), then by (2.5) we get
\[
\|\langle m_j(f_1) + m_j(f_2) - m_j(f_3)\rangle\alpha\| \leq 2^{-(j-1)2^{-2j}\min(C(f_1), C(f_2), C(f_3))},
\] (2.23)
and so by Lemma 2 (ii) and (2.20) we get
\[
\sum_{n \in H_3} \max_{\alpha \in K} l^{1+\epsilon} (\|n\alpha\|) \leq m \Sigma < \infty.
\] (2.24)
If \( N \geq 1 \) and \( \alpha \in K \), then
\[
\|(m_1(g_N) - 1)\alpha\| \leq \|m_1(g_N)\alpha - g_N(\alpha)\| + \|g_N(\alpha) - \alpha\| \leq 2^{1-N} \quad (2.25)
\]
by (2.1), (2.4) and (2.5), so by the definition of \( H_4 \), we obtain
\[
\sum_{n \in H_4} \max_{\alpha \in K} \|n\alpha\| \leq \sum_{N=1}^{\infty} (1 + \log_2 N) l^{1+\epsilon} \left(2^{1-N}\log_2 N\right) < \infty. \quad (2.26)
\]
The relations (2.10), (2.22), (2.24) and (2.26) prove (2.14).

Now, assume that for a \( \beta \in T \) we have an \( n_0 > 0 \) such that (2.15) is true if \( n \in H \) and \(|n| > n_0\). Since \( K \) is a Kronecker set, so \( \|n\alpha\| > 0 \) for \( 0 \neq n \in \mathbb{Z}, \alpha \in K \). Therefore, we see from (2.21) (and (2.3)) that
\[
0 < |m_{j+1}(f) - m_j(f)| \leq n_0
\]
can hold only for finitely many pairs \( f \in F, \ j \geq 1 \); we see from (2.23) that if \( f_1, f_2, f_3 \in F \) are given with \( f_3 = f_1 + f_2 \), then
\[
0 < |m_j(f_1) + m_j(f_2) - m_j(f_3)| \leq n_0
\]
can hold only for finitely many \( j \geq 1 \); and from (2.25) that
\[
0 < |m_1(g_N) - 1| \leq n_0
\]
can hold only for finitely many \( N \). Then, by Lemma 3, we obtain the following inequalities (using \( H_2 \subseteq H, \ H_3 \subseteq H, \ H_4 \subseteq H \), respectively):
\[
\|(m_{j+1}(f) - m_j(f))\beta\| < \frac{1/10}{2^{j-2+C(f)}} \quad (2.27)
\]
for all but finitely many pairs \( f \in F, \ j \geq 1 \);
\[
\|(m_j(f_1) + m_j(f_2) - m_j(f_3))\beta\| < \frac{1/10}{2^{j-2}} \quad (2.28)
\]
for every triple \( f_1, f_2, f_3 \in F \) with \( f_3 = f_1 + f_2 \) and for large enough \( j \in J_{f_1,f_2,f_3} \);
\[
\|(m_1(g_N) - 1)\beta\| < \frac{1/10}{N/2} \quad (2.29)
\]
for large enough \( N \).

Then from (2.27), for all but finitely many pairs \( f \in F, \ j_1 \geq 1 \) we have
\[
\|(m_{j_2}(f) - m_{j_1}(f))\beta\| < \frac{2/5}{2^C(f)} \sum_{j=j_1}^{j_2-1} 2^{-j} \quad (2.30)
\]
for every \( j_2 > j_1 \). This implies that \( m_j(f)\beta \) is a Cauchy sequence for every \( f \in F \), so
\[
\phi(f) := \lim_{j \to \infty} m_j(f)\beta \quad (2.31)
\]
exists, (2.16) is satisfied for all but finitely many pairs \( f \in F, \ j \geq 1 \) by (2.30), and since every \( J_{f_1,f_2,f_3} \) is an infinite set, \( \phi : F \to T \) is a group.
homomorphism by (2.28) and (2.31). We also see that for large $N$, by (2.16), (2.4) and (2.29), we have

$$\|\phi(g_N) - \beta\| \leq \frac{1}{N}. \quad (2.32)$$

If $(N, a)$ is a fixed pair with $N \geq 0$, $1 \leq a \leq 2^N$ and $K_{N,a} \neq \emptyset$, then

$$\|\phi(f_{N,a,r})\| \leq \left\|\phi(f_{N,a,r}) - m_{j(N,a,r)}(f_{N,a,r}) \beta\right\| + \left\|m_{j(N,a,r)}(f_{N,a,r}) \beta\right\|,$$

and so

$$\lim_{r \to \infty} \sup \|\phi(f_{N,a,r})\| \leq \frac{1}{10}$$

by (2.16), (2.7), using also the assumption on $\beta$, (2.8) and $H_1 \subseteq H$. Then (2.17) follows from Lemma 4, because $F_{N,a}$ is obviously isomorphic to $T^{(2)}$. We now prove (2.18). Assume that $N$ is large and

$$|k_{N,a}| > 2^{R(N)} \quad (2.33)$$

for some $1 \leq a \leq 2^N$. Take an integer $r$ such that

$$2 |k_{N,a}| \leq 2^r \leq 4 |k_{N,a}|. \quad (2.34)$$

Then $r > R(N)$, so $m_{j(N,a,r)}(f_{N,a,r}) \in H_1 \subseteq H$, and so for large $N$ we have (see (2.8)) that

$$\left\|m_{j(N,a,r)}(f_{N,a,r}) \beta\right\| < \frac{1}{10}. \quad (2.35)$$

But (2.34) and (2.17) imply

$$\|\phi(f_{N,a,r})\| \geq \frac{1}{4},$$

which contradicts (2.35) for large $N$ by (2.16) and (2.7). Therefore (2.33) cannot be true for large $N$, so (2.18) is proved. To prove (2.19), if $N \geq 0$, $1 \leq a \leq 2^N$ are arbitrary and $k_{N,a} \neq 0$, which implies $K_{N,a} \neq \emptyset$ by definition, we take an $\alpha_{N,a} \in K_{N,a}$, and then, by the definition of $g_N$ and by the already proved properties of $\phi$, we have

$$\|\phi(g_N) - \sum_{1 \leq a \leq 2^N, k_{N,a} \neq 0} k_{N,a} \alpha_{N,a}\| \leq 2^{-N} \sum_{a=1}^{2^N} |k_{N,a}|,$$

and together with (2.32), this proves (2.19).

\[\square\]

Proof of Theorem 1. For every $N \geq 0$ we take some integer $j(N) \geq 1$ such that the sequence $j(N)$ is strictly increasing and

$$\sum_{N=0}^{\infty} 2^{N-1} \left(R(N) + 2\right)^2 l^{1+\epsilon} \left(2^{-j(N)}\right) < \infty. \quad (2.36)$$

Let

$$U = \left\{(N,a) : N \geq 0, 1 \leq a \leq 2^{N-1}, K_{N,2a-1} \neq \emptyset, K_{N,2a} \neq \emptyset\right\},$$
define $A^* \subseteq \mathbb{Z}^2$ as
\[
A^* = \left\{ \left( m_{j(N)}(f_{N,2a-1,r_1}), m_{j(N)}(f_{N,2a,r_2}) \right) : (N, a) \in U, \quad 1 \leq r_1, r_2 \leq R(N) + 2 \right\},
\]
and let $A = A^* \cup \{(n, n) : n \in H\}$. Note that if $(N, a), (N^*, a^*) \in U$, and $1 \leq r_1 \leq R(N) + 2$, $1 \leq r_1^* \leq R(N^*) + 2$, then
\[
m_{j(N)}(f_{N,2a-1,r_1}) \neq m_{j(N^*)}(f_{N^*,2a^*-1,r_1^*}), \text{ if } (N, a) \neq (N^*, a^*). \quad (2.37)
\]
Indeed, this follows from the fact that $j$ is strictly increasing (so one-to-one), using (2.6) and the definition of $U$.

Assume that $\beta \in T$ satisfies (1.2) for all but finitely many $n = (n_1, n_2) \in A$. Then (2.15) is true for all but finitely many $n \in H$, we can apply Lemma 1. If $N$ is large, and we assume that $k_{N,2a-1} \neq 0$ and $k_{N,2a} \neq 0$ for some $1 \leq a \leq 2^{N-1}$ (this implies $(N, a) \in U$ by the definitions), then by (2.18) we can take a pair $1 \leq r_1, r_2 \leq R(N) + 2$ such that
\[
2|k_{N,2a-1}| \leq 2^{r_1} \leq 4|k_{N,2a-1}|, \quad 2|k_{N,2a}| \leq 2^{r_2} \leq 4|k_{N,2a}|.
\]
Then by (2.17), we have
\[
\| \phi(f_{N,2a-1,r_1}) \| \geq \frac{1}{4}, \quad \| \phi(f_{N,2a,r_2}) \| \geq \frac{1}{4},
\]
and, in view of (2.16), $j(N) \to \infty$, the definition of $A$, (2.37) and the property of $\beta$, this is a contradiction for large $N$. Therefore, if $N$ is large, then $k_{N,2a-1}k_{N,2a} = 0$ for every $1 \leq a \leq 2^{N-1}$, and since clearly $k_{N,2a-1} + k_{N,2a} = k_{N-1,a}$, this easily implies that $\sum_{a=1}^{2^N} |k_{N,a}|$ is constant for large $N$. In view of (2.19) and the compactness of the sets $K_s$, this proves that $\beta \in G$.

Now, let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be given distinct elements of $K$. Then it is clear that if $N$ is large enough ($N \geq N_0$), then for any $1 \leq a \leq 2^{N-1}$ we can take a $\delta(N, a) \in \{0, 1\}$ such that
\[
\alpha_1, \alpha_2, \ldots, \alpha_t \notin K_{N,2a-\delta(N,a)},
\]
i.e.
\[
f_{N,2a-\delta(N,a),r}(\alpha_i) = 0
\]
for every $r \geq 1$, $1 \leq i \leq t$. Then, defining $\delta(N, a) \in \{0, 1\}$ arbitrarily for $0 \leq N < N_0$, $1 \leq a \leq 2^{N-1}$, by (2.5) and (2.36) we have
\[
\sum_{N=0}^{\infty} \sum_{a=1}^{2^{N-1}} \sum_{1 \leq r_1, r_2 \leq R(N)+2} l^{1+\epsilon} \left( \| m_{j(N)}(f_{N,2a-\delta(N,a),r_2-\delta(N,a)}) \|_{\alpha_i} \right) < \infty
\]
for $1 \leq i \leq t$. This, together with (2.14), means that defining $f$ on $A^*$ by
\[
f \left( \left( m_{j(N)}(f_{N,2a-1,r_1}), m_{j(N)}(f_{N,2a,r_2}) \right) \right) = m_{j(N)}(f_{N,2a-\delta(N,a),r_2-\delta(N,a)}),
\]
(the definition is correct by (2.37)), and extending \( f \) to \( A \) by \( f((n,n)) = n \) for \( n \in H \), we have (1.3) for every \( 1 \leq i \leq t \). We proved the existence of such an \( f \) for \( \alpha_1,\alpha_2,\ldots,\alpha_t \in K \), but since \( K \) generates \( G \), such an \( f \) exists also for \( \alpha_1,\alpha_2,\ldots,\alpha_t \in G \), in view of Lemma 2 (i). Then (1.1) follows easily, so the theorem is proved. □

3. Some lemmas

**Lemma 2.** (i) There is a constant \( M > 0 \) such that if \( x, y \geq 0 \), then
\[
 l^{1+\epsilon}(x+y) \leq M(l^{1+\epsilon}(x) + l^{1+\epsilon}(y)).
\]

(ii) There is an \( m > 0 \) constant such that for any \( a > 0 \) we have
\[
 \sum_{r=0}^{\infty} l^{1+\epsilon}(2^{-r-a}) \leq ma^{-\epsilon}.
\]

**Proof.** For statement (i) we may obviously assume that \( 0 < x, y < 1/4 \). Then
\[
 x + y \leq 2 \max(x,y) \leq \sqrt{\max(x,y)},
\]
and so
\[
 l^{1+\epsilon}(x+y) \leq l^{1+\epsilon}(\sqrt{\max(x,y)}) = \left(-\log_2 \left(\sqrt{\max(x,y)}\right)\right)^{(1+\epsilon)}
\]
\[
 = 2^{1+\epsilon} l^{1+\epsilon}(\max(x,y)),
\]
which proves (i). Statement (ii) is trivial from the definitions. □

**Lemma 3.** If \( \omega \in T, k \geq 1 \) is an integer, and
\[
 \|\omega\|, \|2\omega\|, \|4\omega\|, \ldots, \|2^k\omega\| \leq \delta < \frac{1}{10},
\]
then \( \|\omega\| \leq \frac{\delta}{2^k} \).

**Proof.** This is easy, and proved as Lemma 3 of [B-S]. □

**Lemma 4.** If \( \phi : T(2) \to T \) is a group homomorphism and
\[
 \limsup_{r \to \infty} \left\| \phi \left(\frac{1}{2^r}\right)\right\| < \frac{1}{4},
\]
then there is a unique integer \( k \) such that \( \phi(\alpha) = k\alpha \) for every \( \alpha \in T(2) \).

**Proof.** The uniqueness is obvious, we prove the existence. It is well-known that the Pontriagin dual of the discrete group \( T(2) \) is the additive group \( \mathbb{Z}_2 \) of 2-adic integers. Hence there is a 0-1 sequence \( b_r \) \( (r \geq 0) \) such that
\[
 \phi(\alpha) = \left(\sum_{r=0}^{\infty} b_r 2^r\right) \alpha
\]
for every $\alpha \in T^{(2)}$, hence
\[ \phi \left( \frac{1}{2^r} \right) = \frac{b_0}{2^r} + \frac{b_1}{2^{r-1}} + \ldots + \frac{b_{r-1}}{2} \] (3.3)
for every $r \geq 1$. We see from (3.3) that if $b_{r-1} = 1$, $b_{r-2} = 0$, then
\[ \frac{1}{2} \leq \left\{ \phi \left( \frac{1}{2^r} \right) \right\} \leq \frac{3}{4}, \]
which is impossible for large enough $r$, in view of (3.1). Consequently, the sequence $b_r$ is constant for large enough $r$. If this constant is 0, i.e. $b_r = 0$ for $r \geq r_0$, then using (3.2), we get the lemma at once. If the constant is 1, so $b_r = 1$ for $r \geq r_0$, then, since
\[ \sum_{r=0}^{\infty} 2^r = -1 \]
in $\mathbb{Z}_2$, one obtains the lemma from (3.2) with
\[ k = -1 - \left( (1 - b_0) + 2(1 - b_1) + \ldots + 2^{r_0-1}(1 - b_{r_0-1}) \right). \]

\[ \square \]

4. Proof of Theorem 2

If $G$ is a group and $d$ is a metric on $G$, we say that $(G, d)$ is a Polish group, if $d$ is a complete metric, and $G$ with this metric is a separable topological group.

The following proposition essentially appears on p. 541. of [A-N], but since they give only a brief indication of the proof, we think that it is worth to include a proof here.

Proposition 1. Assume that $K$ is an uncountable compact subset of $T$, and $K$ is independent over $\mathbb{Z}$. Let $G \leq T$ be the subgroup generated by $K$. Let $d$ be a metric defined on $G$ such that $(G, d)$ is a Polish group. Then the injection map
\[ i : (G, d) \to T, \quad i(g) = g \text{ for every } g \in G \]
is not continuous (we take on $T$ its usual topology, inherited from $\mathbb{R}$).

Proof. Let $Q$ be a countable dense subgroup in $(G, d)$ (such a subgroup clearly exists, since $(G, d)$ is separable). Consider $Q$ with the discrete topology (discrete metric). Then $(Q, G)$ is a Polish (polonais) transformation group in the sense of [E], moreover, it clearly satisfies Condition C on p. 41. of [E]. Since $Q$ is not locally closed in $G$ by our conditions, condition (5) of Theorem 2.6 of [E] is not satisfied. Hence (9) of that theorem is also false, therefore there is a Borel measure $\mu$ on $G$ with $\mu(G) = 1$ such that
(i) each $Q$-invariant measurable subset of $G$ has measure 0 or 1;
(ii) each point of $G$ has measure 0.
Indeed, $\mu(G) = 1$ can be assumed, since $\mu$ is nontrivial and finite by \([E]\), (i) follows since $\mu$ is ergodic in the sense of \([E]\), and (ii) is true by (i), because $\mu$ is not concentrated in a $Q$-orbit.

The measure $\mu$ then has the following additional property, which is a strengthening of (ii):

(iii) if $F \subseteq G$ is a closed subset (in the $d$-topology) and $\mu(F) > 0$, then there is an $A \subseteq F$ with $0 < \mu(A) < \mu(F)$.

It follows by another application of Theorem 2.6 of \([E]\). Indeed, let $\{0\}$ be the trivial group, then $\{(0), F\}$ is a polonais transformation group satisfying Conditon C on p.41. of \([E]\), (5) of Theorem 2.6 is true, hence (8) of Theorem 2.6, using (ii), gives (iii).

Now, we are able to prove the proposition. Assume that $i : (G, d) \to T$ is continuous, and we will get a contradiction. For $t \geq 1, n_1, n_2, \ldots, n_t \in \mathbb{Z}$ set

$$E(n_1, n_2, \ldots, n_t) = \{n_1x_1 + n_2x_2 + \ldots + n_tx_t : x_1, x_2, \ldots, x_t \in K\}.$$ 

Every $E(n_1, n_2, \ldots, n_t)$ is a closed set in $(G, d)$, since it is closed in $T$ and $i$ is continuous. Since $G = \bigcup_{t \geq 1} \bigcup_{n_1, n_2, \ldots, n_t \in \mathbb{Z}} E(n_1, n_2, \ldots, n_t),$

hence $\mu(E(n_1, n_2, \ldots, n_t)) > 0$ for some values of the parameters.

Let $g \in G, t \geq 1, n_1, n_2, \ldots, n_t \in \mathbb{Z}$ be minimal with the property that $\mu(g + E(n_1, n_2, \ldots, n_t)) > 0,$

in the sense that

$$\mu(h + E(m_1, m_2, \ldots, m_r)) = 0$$

for every $h \in G, r \geq 1, m_1, m_2, \ldots, m_r \in \mathbb{Z}$ with

$$|m_1| + |m_2| + \ldots + |m_r| + |r| < |n_1| + |n_2| + \ldots + |n_t| + |t|.$$  \hfill (4.2)

By (iii), writing $F = g + E(n_1, n_2, \ldots, n_t)$, there is an $A \subseteq F$ with $0 < \mu(A) < \mu(F)$. Then $\mu\left(\bigcup_{q \in Q} (q + A)\right) > 0$, hence $\mu\left(\bigcup_{q \in Q} (q + A)\right) = 1$ by (i). We prove that

$$\mu\left(\left(\bigcup_{q \in Q} (q + A)\right) \cap (F \setminus A)\right) = 0.$$ 

This will give a contradiction, because $\mu(F \setminus A) > 0$. Since $Q$ is countable, it is enough to prove that $\mu((q + A) \cap F) = 0$ for every $0 \neq q \in Q$, which follows, if we prove

$$\mu((q + F) \cap F) = 0$$ \hfill (4.3)

for every $0 \neq q \in Q$. 

Assume that \( q + f_1 = f_2, f_1 = g + e_1, f_2 = g + e_2 \), where \( f_1, f_2 \in F, e_1, e_2 \in E(n_1, n_2, \ldots, n_t) \). For \( i = 1, 2 \) let
\[
e_i = n_1 x_{i1} + n_2 x_{i2} + \ldots n_t x_{it}
\]
with \( x_{ij} \in K \) for \( i = 1, 2, 1 \leq j \leq t \). Let
\[
q = \nu_1 x_{01} + \nu_2 x_{02} + \ldots \nu_s x_{0s}
\]
with \( s \geq 1 \), and \( \nu_i \in \mathbb{Z}, x_{0l} \in K \) for \( 1 \leq l \leq s \). Since \( q + e_1 = e_2, q \neq 0 \), and \( K \) is independent over \( \mathbb{Z} \), there are integers \( 1 \leq i \leq 2, 1 \leq j \leq t \) and \( 1 \leq l \leq s \) such that \( x_{ij} = x_{0l} \). Therefore, if
\[
E := \bigcup_{1 \leq l \leq s} \bigcup_{m \in \mathbb{Z}} (mx_{0l} + E(m_1, m_2, \ldots, m_r)),
\]
where
\[
H := \{(r, m_1, m_2, \ldots, m_r) : r \geq 1, m_1, m_2, \ldots, m_r \in \mathbb{Z}, (4.2) \text{ is true}\},
\]
then \( e_i \in E \) for some \( 1 \leq i \leq 2 \). Hence
\[
f_2 \in (g + E) \cup (g + q + E).
\]
Since \( \mu(g + E) = \mu(g + q + E) = 0 \) by (4.1), (4.2), so (4.3) is true, and the proposition is proved.

**Proof of Theorem 2.** Assume that
\[
G = \left\{ \beta \in T : \lim_{n \in A} \|n\beta\| = 0 \right\}
\]
for some infinite \( A \subseteq \mathbb{Z} \). For \( x, y \in G \) let
\[
d(x, y) = \|x - y\| + \max_{n \in A} \|n(x - y)\|. \quad (4.4)
\]
It is clear that \( d \) is a metric on \( G \), and \( (G, d) \) is a topological group. We show that \( d \) is complete. Let \( \beta_j \in G, j \geq 1 \) be a Cauchy sequence with respect to \( d \). Then \( \beta_j \) is a Cauchy sequence also in \( T \) by (4.4), so there is a \( \beta \in T \) such that \( \|\beta_j - \beta\| \to 0 \) as \( j \to \infty \). Now, for \( n \in A, j_1, j_2 \geq 1 \) we have
\[
\|n(\beta_{j_1} - \beta)\| \leq \|n(\beta_{j_1} - \beta_{j_2})\| + \|n(\beta_{j_2} - \beta)\|. \quad (4.5)
\]
Letting \( j_2 \to \infty \) for fixed \( n \) and \( j_1 \) we get
\[
\|n\beta\| \leq \|n\beta_{j_1}\| + \limsup_{j_2 \to \infty} d(\beta_{j_1}, \beta_{j_2}),
\]
and \( \beta_{j_1} \in G \) gives
\[
\limsup_{n \in A} \|n\beta\| \leq \limsup_{j_2 \to \infty} d(\beta_{j_1}, \beta_{j_2})
\]
for every $j_1 \geq 1$, which proves $\beta \in G$. Let $\epsilon > 0$, then we can take $j_2, N \geq 1$ so that
\[
\| n (\beta_{j_2} - \beta) \| + \sup_{j_1 \geq j_2} d(\beta_{j_1}, \beta_{j_2}) < \epsilon
\]
for every $n \in A$, $|n| \geq N$. Hence for $j_1 \geq j_2$, $n \in A$, $|n| \geq N$ we have $\| n (\beta_{j_1} - \beta) \| < \epsilon$ by (4.5). Since for any fixed $|n| < N$ we know that $\| n (\beta_{j_1} - \beta) \| \to 0$ as $j_1 \to \infty$, this proves $d(\beta_{j_1}, \beta) \to 0$, so $d$ is complete.

Let $X$ be a countable dense subset in $T$, and for $N, l \geq 1$ integers, $x \in X$ let
\[
U_{N,l,x} = \left\{ \beta \in G : \| \beta - x \| + \max_{n \in A, |n| \leq N} \| n (\beta - x) \| + \max_{n \in A, |n| > N} \| n \beta \| < \frac{1}{l} \right\}.
\]
It is easy to check that if we take an element from each nonempty $U_{N,l,x}$, then we get a countable dense subset of $(G, d)$. So the conditions of Proposition 1 are satisfied, hence $i : (G, d) \to T$ is not continuous. But this contradicts (4.4), so the theorem is proved. \hfill \Box

5. Some remarks and problems

If $K$ is finite, it follows from [Bi2], Theorem 1 (ii) that Theorem 1 of the present paper would be false for $\epsilon = 0$. But we cannot decide the following

**Problem 1.** Let $K$ be uncountable. Is Theorem 1 true with $\epsilon = 0$?

The following proposition is a consequence of [V], p.140, Theorem 2' (the quoted theorem of Varopoulos is stronger than this statement):

**Proposition 2.** Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$, then there is an infinite subset $A \subseteq \mathbb{Z}$ such that
\[
G = \left\{ \beta \in G \cup L : \lim_{n \in A} \| n \beta \| = 0 \right\}.
\]

Compare Proposition 2 with our Theorem 2. We do not know whether Proposition 2 can be strengthened in the following way:

**Problem 2.** Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$. Is there an infinite subset $A \subseteq \mathbb{Z}$ such that
\[
G = \left\{ \beta \in G \cup L : \lim_{n \in A} \| n \beta \| = 0 \right\},
\]
and
\[
\sum_{n \in A} \| n \alpha \| < \infty
\]
for every $\alpha \in G$?

We state without proof our following partial result in this direction.
Theorem 3. Let $L \subseteq T$ be a compact set with $L \cap G = \emptyset$, and let $v$ be a strictly increasing continuous function on the interval $[0,1/2]$ with $v(0) = 0$. Then there is an infinite subset $A \subseteq \mathbb{Z}$ such that we have
\[ \sum_{n \in A} l^{1+\epsilon} (\|n\alpha\|) < \infty \]
for every $\alpha \in G$, but
\[ \sum_{n \in A} v (\|n\beta\|) = \infty \]
for every $\beta \in L$.

Remark that this theorem implies at once the result mentioned on p.40. of [H-M-P], namely that $G$ is a saturated subgroup of $T$ (for the definition of a saturated subgroup, see [H-M-P] or [N], Ch. 14). We note that the above-mentioned Theorem 2’ on [V], p.140, also implies that $G$ is saturated.

Finally, we mention that Theorem 2 and Proposition 2 together show that if $K$ is uncountable, then $G$ is a g-closed but not basic g-closed subgroup of $T$ in the terminology of [D-M-T]. This answers the question of D. Dikranjan (oral communication) about the existence of such subgroups of $T$.

References

[Bi2] A. Biró, Characterizing sets for subgroups of compact groups II.: the general case. Preprint, 2004