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Configurations of rank-40\(r\) extremal even unimodular lattices (\(r = 1, 2, 3\))

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Résumé. Nous montrons que, si \(L\) est un réseau unimodulaire pair extrémal de rang 40\(r\) avec \(r = 1, 2, 3\), alors \(L\) est engendré par ses vecteurs de normes 4\(r\) et 4\(r\) + 2. Notre résultat est une extension de celui d’Ozeki pour le cas \(r = 1\).

Abstract. We show that if \(L\) is an extremal even unimodular lattice of rank 40\(r\) with \(r = 1, 2, 3\), then \(L\) is generated by its vectors of norms 4\(r\) and 4\(r\) + 2. Our result is an extension of Ozeki’s result for the case \(r = 1\).

1. Introduction

A lattice of rank \(n\) is a free \(\mathbb{Z}\)-module of rank \(n\) equipped with a positive-definite inner product \((\cdot, \cdot) : L \times L \to \mathbb{R}\). The dual of \(L\), denoted \(L^*\), is the set

\[ L^* = \{ y \in L \otimes \mathbb{R} : \forall x \in L, (x, y) \in \mathbb{Z} \}, \]

which itself forms a lattice of the same rank as \(L\). For a lattice vector \(x \in L\), we call \((x, x)\) the norm of \(x\). A lattice \(L\) is integral if \((x, x') \in \mathbb{Z}\) for all \(x, x' \in L\), i.e. if and only if \(L \subseteq L^*\). An integral lattice is said to be unimodular if it is self-dual (\(L = L^*\)).

A lattice \(L\) is called even if and only if every lattice vector has an even integer norm, i.e. \((x, x) \in 2\mathbb{Z}\) for \(x \in L\). An even lattice is automatically integral by the familiar parallelogram identity,

\[ 2(x, x') = (x + x', x + x') - (x, x) - (x', x'). \]

Lattices that are simultaneously even and unimodular are especially rare. Indeed, such a lattice’s rank must be divisible by 8. Sloane proved that if \(L\) is an even unimodular lattice of rank \(n\) then the minimal (nonzero) norm in \(L\) is bounded by

\[ \min_{\substack{x \in L \\setminus \{0\}}} (x, x) \leq 2 \lfloor n/24 \rfloor + 2 \]

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Mots clefs. Even unimodular lattices, extremal lattices, weighted theta series.
An even unimodular lattice of rank $n$ is called extremal if it attains the bound (1.1).

Ozeki [6, 8] showed that if $L$ is an extremal even unimodular lattice of rank 32 or 48 then $L$ is generated by its vectors of minimal norm. The first author [5] showed analogous results for extremal even unimodular lattices of ranks 56, 72, and 96. In a similar vein, Ozeki [7] showed that if $L$ is extremal even unimodular of rank 40, then $L$ is generated by its vectors of norms 4 and 6. Here, we extend and slightly simplify Ozeki’s methods, recovering Ozeki’s rank-40 result and obtaining analogous results for extremal even unimodular lattices of ranks 80 and 120.

2. Modular forms and theta series

We will use the notation $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ for the upper half plane of complex numbers. A modular form of weight $k$ for the group $\text{PSL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathcal{H} \to \mathbb{Z}$ which is holomorphic at $i\infty$ and satisfies

$$f \left( \frac{az + b}{cz + d} \right) = (cz + d)^k f(z)$$

for all $(a \ b \ c \ d) \in \text{PSL}_2(\mathbb{Z})$. If a modular form $f$ vanishes at $z = i\infty$, it is called a cusp form.

Let $M_k$ and $M_0^k$ be the $\mathbb{C}$-vector spaces of modular forms and cusp forms of weight $k$ respectively. It is known that the Eisenstein series $E_4(z) = 1 + 240e^{2\pi i z} + 2160e^{4\pi i z} + 6720e^{6\pi i z} + \cdots$ and $E_6(z) = 1 - 504e^{2\pi i z} - 16632e^{4\pi i z} - 122976e^{6\pi i z} - \cdots$, which are modular forms of weights 4 and 6 respectively, freely generate the spaces $M_k$ in the sense that any nonzero modular form can be written uniquely as a weighted homogeneous polynomial in $E_4$ and $E_6$. This implies that $\dim(M_k) = 0$ for $k$ odd, negative, or $k = 2$; that $\dim(M_{2k}) = 1$ and $\dim(M_{2k-2}) = 0$ for $k = 0, 2 \leq k \leq 5$ and $k = 7$; and that multiplication by the weight-12 modular form $\Delta = 12^{-3}(E_4^3 - E_6^2)$ defines an isomorphism $M_{k-12} \cong M_k^0$. More information on the theory of modular forms for $\text{PSL}_2(\mathbb{Z})$ can be found in [9].

The theta function $\Theta_L : \mathcal{H} \to \mathbb{Z}$ associated to a lattice $L$ is defined by

$$\Theta_L(z) = \sum_{x \in L} e^{\pi i (x,x)z},$$

it is a generating function encoding the norms of $L$’s vectors. For a homogeneous harmonic polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$, i.e. a homogeneous polynomial for which $\sum_{j=1}^n \frac{\partial^2 P}{\partial x_j^2} \equiv 0$, we define the weighted theta series $\Theta_{L,P}$ by

$$\Theta_{L,P}(z) = \sum_{x \in L} P(x)e^{\pi i (x,x)z}.$$
As shown in [9, 3], if $L$ is an even unimodular lattice of rank $n$ then $\Theta_L$ is a modular form of weight $\frac{n}{2}$, and if in addition $P$ is a homogeneous harmonic polynomial of degree $d$, then $\Theta_{L,P}$ is a modular form of weight $\frac{n}{2} + d$.

3. Main result

We denote by $P_{d,x_0}(x)$ the “zonal spherical harmonic polynomial” of degree $d$, related to the Gegenbauer polynomial by

$$P_{d,x_0}(x) = G_d((x, x_0), ((x, x) (x_0, x_0))^{1/2}),$$

where $G_d(\cdot, \cdot)$ is the homogeneous polynomial of degree $d$ such that $G_d(t, 1)$ is the Gegenbauer polynomial of degree $d$ evaluated at $t$ [1].

We let $L$ be an extremal even unimodular lattice of rank $40r$ (where $r \in \{1, 2, 3\}$), and adopt the notation used by Ozeki in [7]: For an even unimodular lattice $L$, we denote by $\Lambda_{2m}(L)$ the set of vectors in $L$ having norm $2m$. We denote by $\mathcal{L}_{2m}(L)$ the sublattice of $L$ generated by $\Lambda_{2m}(L)$, and similarly denote by $\mathcal{L}_{2m_1+2m_2}(L)$ the sublattice of $L$ generated by $\Lambda_{2m_1}(L) \cup \Lambda_{2m_2}(L)$.

We define $a(2k, L) = |\Lambda_{2k}(L)|$. It is clear that the theta series $\Theta_L$ is given by $\Theta_L(z) = \sum_{k=0}^{\infty} a(2k, L) e^{2k \pi z}$. We note that

$$4r = 2[5r/3] + 2 = \min\{2k > 0 : a(2k, L) \neq 0\}$$

is the minimal norm of vectors in $L$ and use the notation

$$N_j(x) = |\{y \in \Lambda_{4r}(L) : (x, y) = j\}|,$$

$$M_j(x) = |\{y \in \Lambda_{4r+2}(L) : (x, y) = j\}|.$$

Using the involution $y \mapsto -y$ of $\Lambda_m(L)$, we see that we have $N_j(x) = N_{-j}(x)$ and $M_j(x) = M_{-j}(x)$ for any $j \in \mathbb{R}$ and $x \in L \otimes \mathbb{R}$.

We will show the following configuration result, which directly extends Ozeki’s [7] result for extremal even unimodular lattices of rank 40:

**Theorem 3.1.** For $r = 1, 2, 3$ and $L$ extremal even unimodular of rank $40r$, we have $L = \mathcal{L}_{4r+(4r+2)}(L)$.

**Proof.** We partition $L$ into its equivalence classes modulo $\mathcal{L}_{4r+(4r+2)}(L)$. We need only show that any class $[x] \in L/\mathcal{L}_{4r+(4r+2)}(L)$ is represented by a vector $x_0 \in [x]$ with $(x_0, x_0) \leq 4r + 2$.

Now, we suppose there exists some equivalence class $[x_0] \in L/\mathcal{L}_{4r+(4r+2)}(L)$ where $x_0 \neq 0$ is a representative of minimal norm with $(x_0, x_0) = 2t$ for some $t \geq 2r + 2$. We have the inequality

$$|\langle x_0, x \rangle| \leq 2r$$

for all $x \in \Lambda_{4r}(L)$.

Indeed, if $(x_0, \pm x) > 2r$, then $L$ contains a vector $x \neq x_0$ with norm

$$(x \mp x_0, x \mp x_0) = (x, x) \mp 2(x, x_0) + (x_0, x_0) < (x_0, x_0),$$
contradicting the minimality of \( x_0 \).

Similarly, we have

\[ |(x_0, x)| \leq 2r + 1 \text{ for all } x \in \Lambda_{4r+2}(L). \]

From (3.2) and (3.3), we have the equations

\[ \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2k} = \sum_{j=1}^{2r} 2 \cdot j^{2k} \cdot N_j(x_0), \]

\[ \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2k} = \sum_{j=1}^{2r+1} 2 \cdot j^{2k} \cdot M_j(x_0), \]

for all \( k > 0 \).

We extract from the theta series \( \Theta_L \) of \( L \) the coefficients \( a(4r, L) \) and \( a(4r + 2, L) \). We observe immediately from (3.4) and (3.5) that

\[ \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^0 = a(4r, L), \]

\[ \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^0 = a(4r + 2, L). \]

Since \( L \) is even unimodular of rank \( 40r \), we have \( \Theta_{L, P_{d, x_0}} \in M_{20r+d}^0 \) for any \( d > 0 \). By comparing power-series coefficients, we then observe

\[ \Theta_{L, P_{d, x_0}} \equiv 0 \text{ for } d \in \{2, \ldots, 4r - 2, 4r + 2\}, \]

\[ \Theta_{L, P_{4r, x_0}} \equiv c_1 \Delta^{2r} \text{ for a constant } c_1, \]

\[ \Theta_{L, P_{4r+4, x_0}} \equiv c_2 E_4 \Delta^{2r} \text{ for a constant } c_2. \]

From (3.8), we obtain the equations

\[ \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2d} = a(4r, L) \frac{1 \cdot 3 \cdots (2d - 1)}{40r \cdot (40r + 2) \cdots (40r + 2d - 2)} (8r)^d \eta^d, \]

and

\[ \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2d} = \]

\[ a(4r + 2, L) \frac{1 \cdot 3 \cdots (2d - 1)}{40r \cdot (40r + 2) \cdots (40r + 2d - 2)} (8r + 4)^d \eta^d, \]
for \( d \in \{2, \ldots, 4r - 2, 4r + 2\} \). We obtain from (3.9)

\[
(3.13) \quad \sum_{x \in \Lambda_{4r+2}(L)} P_{4r,x_0}(x) = c_{4r} \sum_{x \in \Lambda_{4r}(L)} P_{4r,x_0}(x),
\]

where \( \Delta^{4r} = e^{(4r)\pi iz} + c_{4r}e^{(4r+1)\pi iz} + O(e^{(4r+2)\pi iz}) \). Similarly, (3.10) gives

\[
(3.14) \quad \sum_{x \in \Lambda_{4r+4}(L)} P_{4r+4,x_0}(x) = c_{4r+4} \sum_{x \in \Lambda_{4r+4}(L)} P_{4r+4,x_0}(x),
\]

where \( E_{4r} = e^{(4r)\pi iz} + c_{4r+4}e^{(4r+1)\pi iz} + O(e^{(4r+2)\pi iz}) \).

Combining the equations (3.6), (3.7), (3.11), (3.12), (3.13), and (3.14) with (3.4) and (3.5), we obtain a system of \( 4r + 4 \) homogeneous linear equations in the \( 4r + 3 \) unknowns

\[ N_0(x_0), \ldots, N_{2r}(x_0), M_0(x_0), \ldots, M_{2r+1}(x_0). \]

At this stage, we diverge from our natural generalization of Ozeki’s original methods and obtain the (extended) determinants of these inhomogeneous linear systems; these determinants must vanish because the system is overdetermined.

For \( r = 1, 2, 3 \), these determinants are respectively

\[
(3.15) \quad 2^{55}3^75^{16}7^{13}11^413^419^623^3 \cdot (t - 2) \cdot t \cdot (6t - 13) \cdot (10t^2 - 55t + 77),
\]

\[
(3.16) \quad 2^{132}3^{27}5^{16}7^{10}11^613^{10}23^441^843^647^3 \cdot (t - 4) \cdot t \cdot Q_2(t),
\]

\[
(3.17) \quad 2^{244}3^{48}5^{26}7^{13}11^713^717^623^431^{11}37^559^{14}61^{11}67^571^373^2 \cdot (t - 6) \cdot t \cdot Q_3(t),
\]

where \( Q_2(t) \) is the irreducible quintic

\[
10768t^5 - 242280t^4 + 2202310t^3 - 10101795t^2 + 23361877t - 21771246.
\]

and \( Q_3(t) \) is the irreducible septic

\[
19989882674056909935t^7 - 892881426107875310430t^6
+ 1725803601222654151533t^5 - 187053310321121903406075t^4
+ 122739824990822918423784t^3 - 4874010945909263810320032t^2
+ 10840974078436271024624064t - 1041452776992313369099080.
\]

In each case, there are no integer solutions \( t \geq 2r + 2 \). However, we had assumed the existence of an equivalence class

\[ [x_0] \in \mathcal{L}/\mathcal{L}_{4r+(4r+2)}(L) \]

with minimal-norm representative \( x_0 \neq 0 \) having \( (x_0,x_0) = 2t \) for integral \( t \geq 2r + 2 \); since no such \( t \) exists, all equivalence classes must be generated by vectors having norms \( 4r \) and \( 4r + 2 \) \( \square \).
4. Concluding remarks

A quick inspection will show that our results are the only possible immediate extensions of Ozeki’s methods. In the cases \( r \geq 4 \), it is not possible to extract sufficiently many linear conditions by these exact techniques, as the dimensions of the relevant spaces of cusp forms grow too large.

However, using different analysis, Elkies [4] has shown a stronger result than our Theorem 3.1 in the \( r = 3 \) case: If \( L \) is an extremal even unimodular lattice of rank 120 then \( L = L_{12}(L) \). This result for rank-120 lattices is analogous to Ozeki’s [6, 8] results in dimensions 32 and 48, and to the first author’s [5] results in dimensions 56, 72, and 96.

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