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Galois orbits and equidistribution: Manin-Mumford and André-Oort.

par Andrei YAFAEV

Résumé. On passe en revue une approche unifiée aux conjectures de Manin-Mumford et d'André-Oort basée sur la combinaison de techniques galoisiennes et ergodiques. Ce texte est basé sur les travaux récents de Klingler, Ullmo et Yafaev sur la conjecture de André-Oort, et de Ratazzi et Ullmo sur la conjecture de Manin-Mumford.

Abstract. We overview a unified approach to the André-Oort and Manin-Mumford conjectures based on a combination of Galois-theoretic and ergodic techniques. This paper is based on recent work of Klingler, Ullmo and Yafaev on the André-Oort conjecture, and of Ratazzi and Ullmo on the Manin-Mumford conjecture.

1. Introduction.

This text is based on a talk given by the author at the 25th Journées Arithmetiques held at Edinburgh University in July 2007. The purpose of this note is to outline the proofs of the Manin-Mumford and the André-Oort conjectures based on the "Galois theory/Ergodic theory alternative" in the geometry of abelian (resp. Shimura) varieties. The proof of the Manin-Mumford conjecture outlined here is due to Ratazzi and Ullmo and the proof of the André-Oort conjecture is due to Klingler, Ullmo and the author of the present note. We will give the ingredients of the proofs and sketch the final arguments without going into details. For details we will refer to the papers [5], [7] and [4]. We also recommend the note [6] in which we make our strategy work in the case of subvarieties of products of modular curves.

Let S be an abelian or a Shimura variety. Then S contains a large class of so-called special subvarieties. Precise definitions will be given later on. For now, we just mention that special subvarieties are subvarieties having similar geometric structure to that of the variety S itself. Special points are special subvarieties of dimension zero. Given a special subvariety Z of S, special points contained in Z are dense for the Zariski (and even Archemedian) topology. The André-Oort and Manin-Mumford conjectures assert the converse: a subvariety Z of S containing a Zariski dense set of special points is special.

The strategy of the proofs presented here is as follows. Let S be a Shimura or an abelian variety and let Z be an (absolutely, say) irreducible subvariety containing a Zariski dense set of special points. General theory shows that everything is defined over some number field F. The variety S carries a large class of correspondences that we denote by T, they are Hecke correspondences in the Shimura case and simply multiplications by the integers in the abelian case. First one establishes a criterion stating that Z is special if and only if, roughly speaking, Z stable by some suitable correspondence T.

Now suppose that Z contains a Zariski dense sequence (Z_n) of special subvarieties of dimension d. Of course our initial data is that Z contains a Zariski dense sequence of special subvarieties with $d = 0$. If we could show that Z is either automatically special or contains a Zariski dense set of special subvarieties of dimension strictly larger than d then we would have finished.

The central idea in the proof is the following alternative: consider a sequence (Z_n) of special subvarieties of S. Then one of the following occurs:

(1) The degree of the variety $\mathrm{Gal}(\overline{F}/F) \cdot Z_n$ goes to infinity with n.
(2) The subvarieties Z_n are equidistributed in the following sense. Let μ_n be the probability measure canonically attached to Z_n. There exists a special subvariety Z and a subsequence n_k such that

$$\mu_{n_k} \longrightarrow \mu_Z$$

where μ_Z is the canonical probability measure attached to Z. Furthermore, $Z_{n_k} \subset Z$ for all k large enough.

Which case of the alternative occurs depends on the geometric nature of special subvarieties Z_n.

Let us now come back to our original situation. Let Z be an irreducible subvariety of S containing a Zariski-dense sequence (Z_n) of special subvarieties of dimension d. If the second case of the alternative occurs, then Z is clearly automatically special.

If the first case of the alternative occurs, then using the explicit description of the Galois action on special subvarieties, explicit lower bounds for Galois orbits and the geometric characterisation of special subvarieties, one produces a Zariski dense sequence (Z'_n) of special subvarieties of Z of dimension strictly larger than d.

Let us mention here that there are similar statements in the case of tori and semi-abelian varieties to which the strategy presented above should also apply. Note also that there exists analogues of the André-Oort conjecture in the function fields case in which case the ambient variety is a Drinfeld modular variety. We refer to [1] for this.

The author is very grateful to the organisers of the conference for inviting him and giving him the opportunity to give a lecture.

2. Preliminaries.

In this section we review the relevant definitions. Let us first give definitions in the abelian case.

Definition 2.1. *Let S be an abelian variety over $\mathbb{C}$. Let Z be an irreducible subvariety. Then Z is called special if there exists an abelian subvariety B of S and a torsion point x such that $Z = x + B$ (Z is a translate of an abelian subvariety by a torsion point).*

A point x is special if x is a special subvariety of dimension zero i.e. x is a torsion point.

The fact that special points lying on a special subvariety are (analytically) dense is a simple consequence of the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Let us now turn to the Shimura case. We very briefly recall the definitions concerning Shimura varieties.

Definition 2.2. *A Shimura datum is a pair (G, X) where G is a reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-orbit of an element $h \in \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ (where $\mathbb{S}$ is the restriction of scalars to $\mathbb{R}$ of $\mathbb{G}_{\mathrm{m}\mathbb{C}}$) satisfying Deligne's conditions (see* [3]*).*

Deligne's conditions imply that the connected components of X are Hermitian symmetric domains. Let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$, then consider the set

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K$$

where $G(\mathbb{Q})$ acts on the left on both factors and K acts on the right on the second. This is a, in general, non-connected and singular analytic variety. It is also known (Baily-Borel) that this variety is a quasi-projective algebraic variety. Further, Shimura, Deligne, Milne, Shih, Borovoi constructed canonical models of this variety over some explicitly defined (in terms of G and X) number fields.

The connected components of $\mathrm{Sh}_K(G, X)$ are of the form $\Gamma\backslash X^+$ where X^+ is a connected component of X and $\Gamma \subset G(\mathbb{Q})$ is a certain congruence subgroup. Let S_0 be a component of $\mathrm{Sh}_K(G, X)$. Writing X^+ as $G(\mathbb{R})^+/K_\infty$ where $G(\mathbb{R})^+$ is the neutral component of $G(\mathbb{R})$ and K_∞ is a maximal compact subgroup of $G(\mathbb{R})^+$, we see that

$$S_0 = \Gamma\backslash G(\mathbb{R})^+/K_\infty$$

We notice an analogy with the abelian case. Indeed, if A is an abelian variety of dimension n, $A(\mathbb{C})$ is a complex torus, hence can be written as

$$A(\mathbb{C}) = \Gamma\backslash\mathbb{R}^{2n} = \Gamma\backslash G(\mathbb{R})$$

where G this time is the group $\mathbb{G}_{\mathrm{a}}^{2n}$ and Γ is a lattice in $G(\mathbb{R})$. The Deligne's conditions in the Shimura case are analogous to the existence of a complex structure and a polarisation in the abelian case. We also notice that one of the main differences between Shimura and abelian cases lies in the fact that in the first case the group is reductive and in the second case the group is unipotent. Let us point out here that there is a notion of a *mixed* Shimura variety of which Shimura and abelian varieties are special cases. There are also generalisations of the Manin-Mumford and André-Oort conjectures to this mixed case. We refer the reader to [8] and [9] for this. We will not go into details on this in the present paper. To the author's best knowledge, these conjectures remain open.

Before giving the definition of special subvarieties and points, we define the notion of Hecke correspondence. This is analogous to translation by the group law in the abelian case.

Definition 2.3. *Let g be an element of $G(\mathbb{A}_{\mathrm{f}})$. Let S_g be the Shimura variety $\mathrm{Sh}_{K\cap gKg^{-1}}(G,X)$. Then S_g has two maps to S : π_1 induced by the inclusion $K\cap gKg^{-1}\subset K$ and π_2 induced by the inclusion preceded by right multiplication by g. The Hecke correspondence T_g is defined by*

$$T_g(x) = \pi_2\pi_1^{-1}(x)$$

for $x\in S$.

Definition 2.4. *An irreducible subvariety Z of S is called special if there exists a sub-Shimura datum $(H,X_H)\subset(G,X)$ and $g\in G(\mathbb{A}_{\mathrm{f}})$ such that Z is an irreducible component of $T_g(\mathrm{Sh}_{K\cap H(\mathbb{A}_{\mathrm{f}})})(H,X_H))$ (where, by abusing notation, we identify $\mathrm{Sh}_{K\cap H(\mathbb{A}_{\mathrm{f}})}(H,X_H)$ with its image in S.*

One notices that a special subvariety is a 'translate of a Shimura subvariety by a Hecke correspondence' just like a special subvariety in the abelian case is a translate of an abelian subvariety by the group law.

A special point is a special subvariety of dimension zero. By looking at the Shimura data we notice that this means that H is a torus, in which case X_H is just a point. The fact that special points contained in a special subvariety are dense follows from two facts: one is that a special subvariety always contains a special point x and two that $H(\mathbb{Q})^+$ is dense in $H(\mathbb{R})^+$ (neutral component of $H(\mathbb{R})$.) One simply considers the "orbit" $H(\mathbb{Q})\cdot x$. This orbit consists of special points and is dense.

We are now ready to state the Manin-Mumford and the André-Oort conjecture.

Conjecture 2.5 (Manin-Mumford, André-Oort)**.** *Let S be an abelian or a Shimura variety. Let Z be an irreducible subvariety containing a Zariski-dense set of special points. Then Z is a special subvariety.*

The Manin-Mumford conjecture has been proved for a long time. Actually, some far reaching generalisations of this conjecture (Bogomolov conjecture, Mordell-Lang conjecture) have been established. We recommend a survey by Tzermias ([10]) for details on this.

We just make one last remark about a presentation of special subvarieties in the normalised form. The problem is that a presentation of a special subvariety in the abelian case $Z = x + B$ is not unique as x is defined up to adding a torsion point of B. Similarly, the Shimura subdatum $(H, X_H) \subset (G, X)$ defining a special subvariety in the Shimura case is defined up to multiplying H by a subgroup of the centre of G.

In the abelian case the remedy is the following theorem of Bertrand (an effective form of Poincaré lemma, see [5], Prop 5.1): let B be an abelian subvariety of S (abelian), then there exists an abelian subvariety B' such that

$$S = B + B'$$

and $|B \cap B'|$ is uniformly (in terms of S only) bounded. We now let $Z = x+B$ be a special subvariety. We say that this notation is normalised if $x \in B'$. This definition gives meaning to saying that a sequence $Z_n = x_n + B_n$ (normalised notation) is such that $\mathrm{ord}(x_n)$ is bounded. As we will see,such sequences have the equidistribution property.

Let us now turn to the Shimura case. Again, this case is completely analogous. Let Z be a special subvariety defined by a sub-Shimura data $(H, X_H) \subset (G, X)$. We say that this data is normalised if H is the so-called generic Mumford-Tate group of X_H. This means that H is the smallest subgroup of G such that every $x \colon \mathbb{S} \longrightarrow G_{\mathbb{R}}$ factors through $H_{\mathbb{R}}$. With this definition, again, it makes sense to say that for a sequence (Z_n) of special subvarieties defined by normalised Shimura data (H_n, X_n), there exists a subtorus T of G such that for all n, $H_n = TH_n^{\mathrm{der}}$ (H_n^{der} is the derived subgroup). In the next section we will see that such sequences are equidistributed.

3. Galois orbits and equidistribution, the alternative.

3.1. Equidistribution theorem. Let S be an abelian or Shimura variety. To a special subvariety Z is canonically attached a probability measure μ_Z.

Theorem 3.2. *Suppose S is as before and let Z_n be a sequence of special subvarieties and μ_n the sequence of canonically associated probability measures. We suppose that the following holds.*

- *Suppose that S is abelian, then for each n, Z_n can be written as $Z_n = x_n + B_n$ where the order of x_n is bounded and B_n is an abelian subvariety of S.*
- *Suppose S is a Shimura variety. Then there exists a torus T such that for any Z_n, the Shimura datum (H_n, X_n) defining Z_n satisfies $H_n = TH_n^{\mathrm{der}}$.*

Then the sequence Z_n is equidistributed in the following sense: there exists a special subvariety Z and a subsequence n_k such that μ_{n_k} converges weakly to μ_Z and furthermore $Z_{n_k} \subset Z$ for all k large enough.

The proof of this theorem in the abelian case can be found in section 4 of [5]. It only involves Fourier analysis.

The proof in the Shimura case is much more involved. It can be found in [7], where it is derived from a result of Clozel and Ullmo [2]. The proof involves Ratner's theorem and theorems of Mozes-Shah and Dani-Margulis. We refer to [2] for details and references.

3.3. Galois orbits. The following theorem makes the first case of alternative effective i.e. gives explicit lower bounds for Galois orbits.

Theorem 3.4. *Let S be as above and let (Z_n) be a sequence of special subvarities. Let F be a field of definition of S.*

- *Suppose S is abelian then the following holds. We write $Z_n = x_n + B_n$ in the normalised form. Let $\epsilon > 0$ be a real number. Then there exists a constant $C = C(S, F, \epsilon)$ such that*

  $$\deg(\mathrm{Gal}(\overline{F}/F) \cdot Z_n) \geq C \mathrm{ord}(x_n)^{1-\epsilon}$$

  The degree here is calculated with respect to a fixed projective embedding of S.
- *Assume the GRH for CM fields. Suppose S is a Shimura variety. We make the assumption that K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$. Let (H_n, X_n) be the normalised Shimura datum defining Z_n and let T_n be the connected centre of H_n. Fix an integer $N > 0$. We let K_n be the compact open subgroup $T_n(\mathbb{A}_{\mathrm{f}}) \cap K$ of $T_n(\mathbb{A}_{\mathrm{f}})$. This is a product of compact open subgroups $K_{n,p}$ of $T_n(\mathbb{Q}_p)$. We also let K_n^m be the maximal compact open subgroup of $T_n(\mathbb{A}_{\mathrm{f}})$. This is also a product. For any n we let $i(T_n)$ be the number of primes p such that $K_{n,p} \neq K_{n,p}^m$. We also let d_n be the discriminant of the splitting field of T_n.*

  The statement now is that there exists a constant C (depending on N !) and a constant B, both B and C independent of Z_n, such *that*

$$\deg(\mathrm{Gal}(\overline{F}/F)\cdot Z_n) \geq CB^{i(T_n)}|K_n/K_n^m|(\log(d_n))^N$$

In this case, the degree is taken with respect to the Baily-Borel line bundle on S.

The proof in the abelian case is due to Serre, we refer to [5] for details and references. The proof in the Shimura case can be found in [7] building on an earlier work of the author of this note [11]. One needs, among other things, to give a lower bound for the size of a certain subgroup of the class group of a certain CM field. This is achieved under the assumption of the GRH.

We notice the following : if S is abelian, then the degrees of the Galois orbits go to infinity when $\mathrm{ord}(x_n)$ is unbounded and when it is bounded then the equidistribution property holds. Similarly, in the Shimura case, if T_n varies then one can show that the lower bound above goes to infinity. When T_n is fixed, then the equidistribution holds. This is the alternative mentioned in the introduction.

3.5. Galois action and Hecke correspondences. In this section we explain the relation between the Galois action and multiplication by certain integers (abelian case) or translation by Hecke correspondences (Shimura case).

Proposition 3.6. *Let S be an abelian or a Shimura variety and F a number field over which S admits a canonical model. Let Z be a special subvariety and l a prime number.*

- *Suppose S is abelian and write $Z = x + B$. Suppose that l does not divide* $\mathrm{ord}(x)$*. There exists a constant c depending on S and F only (not on Z) and $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that*

$$[p^c]Z = \sigma(Z)$$

- *Suppose S is a Shimura variety and let (H, X_H) be the Shimura data defining Z and T the connected centre of H. Suppose that l splits T and $K_l^m = K_l$. There exists a constant c depending on S and F only and an element m of $T(\mathbb{Q}_l)$ not contained in a compact subgroup and $\sigma \in \mathrm{Gal}(\overline{F}/F)$ such that*

$$\sigma(Z) \subset T_m Z \text{ and } [K : mKm^{-1} \cap K] < l^c$$

In the abelian case, this theorem is also due to Serre. In the Shimura case, this is the complex multiplication theory of Shimura-Taniyama reformulated by Deligne in the context of Shimura varieties. The existence of a Hecke correspondence T_m of bounded degree is proved in [4].

4. Geometric criterion.

In this section we explain the geometric criterion that will be used in the course of the induction.

Theorem 4.1. *Let S be a Shimura or abelian variety and let Z be an irreducible subvariety.*

- *Suppose that there is an integer $n \geq 2$ such that $[n]Z \subset Z$, then Z is special.*
- *Suppose that S is a Shimura variety. We make a certain number of assumptions. The group G is semisimple of adjoint type, the group K is neat and is a product. The variety Z is Hodge generic (not contained in a proper special subvariety) and contains a special subvariety V defined by a normalised Shimura datum (H, X_H) where H has a non-trivial connected centre T. Suppose that there is a prime number l splitting T and such that the component K_l is contained in an Iwahori subgroup.*

  Suppose that there exists an element m of $T(\mathbb{Q}_l)$ not contained in a compact subgroup such that $Z \subset T_m Z$. Then Z contains a special subvariety V', containing V properly.

The proof of the statement in the abelian case is due to Hindry and is explained in [5]. The proof in the Shimura case can be found in [4]. The proof uses the existence of variations of polarisable $\mathbb{Z}$-Hodge structures over S, a theorem of Yves André's on the Zariski closure of the image of the monodromy, a theorem of Nori's on adelic closure of finitely generated subgroups of $G(\mathbb{Q})$ and some stuff on Bruhat-Tits buildings.

5. The proofs.

In this section we show how the ingredients from the previous sections combine to give proofs of the Manin-Mumford and the André-Oort conjectures. In the Shimura case, the details and verifications are rather tedious, we omit them in what follows and refer the reader to the original papers.

So let S be an abelian or a Shimura variety and Z a subvariety containing a Zariski dense sequence (Z_n) of special subvarieties of dimension d. We suppose that the sequence (Z_n) is such that the second case of the alternative does not occur, otherwise there is nothing to prove. Hence, writing in the normalised form $Z_n = x_n + B_n$ (abelian case), we have $\mathrm{ord}(x_n) \longrightarrow \infty$ or $B^{i(T_n)}|K_n/K_n^m|(\log(d_n))^N \longrightarrow \infty$ (Shimura case). Choose a number field F such that S and Z admit a model over F and Z is irreducible over F.

We let l be a prime number and Z_n a special subavariety such that, in the abelian case

(1) $Z_n = x_n + B_n$ and l does not divide $\mathrm{ord}(x_n)$.
(2) $l \ll \log(\mathrm{ord}(x_n))^A$ where A is some uniform constant.

Then $\sigma Z_n \subset Z \cap [l^c]Z$. If $[l^c]Z \subset Z$, then we are done. Otherwise we replace Z by an F-component of $Z \cap [l^c]Z$ containing Z_n. Therefore the dimension of Z has gone down and the degree multiplied by a uniform power of $\log(\mathrm{ord}(x_n))$. So, either at some point the inclusion $[l^c]Z \subset Z$ has occured or $\dim(Z) = d + 1$. In this last case, the facts that the degree of Z is bounded by a uniform power of $\log(\mathrm{ord}(x_n))$ and that Z contains the Galois orbit of Z_n which is of the size at least a small power of $\mathrm{ord}(x_n)$ force the inclusion $[l^c]Z \subset Z$. This finishes the proof.

The Shimura case is very similar, but technically a lot more complicated. We just outline the proof without going into details. First one has to carry out some reduction steps to put ourselves in the situation where the geometric criterion holds.

- Choose Z_n defined by the Shimura data (H_n, X_n) whose Galois orbit is large.
- We choose a prime l that splits T_n, $K^m_{n,l} = K_{n,l}$ and l is small compared to $B^{i(T_n)}|K_n/K^m_n|(\log(d_n))^N$ (again, smaller then). The choice of l requires the assumption of the GRH (in the form of the effective Chebotarev theorem).

For technical reasons, we also need to assume that the l-component K_l of K is contained in an Iwahori subgoup of $G(\mathbb{Q}_l)$. This is a safe assumption, as replacing K_l by a suitable Iwahori subgroup amounts to passing to a covering of degree bounded by a uniform power of l. We let T_m be the Hecke correspondence of degree bounded by a uniform power of l and such that $\sigma Z_n \subset T_m Z_n$ for some $\sigma \in \mathrm{Gal}(\overline{F}/F)$.

Exactly, as in the abelian case, if $Z \subset T_m Z$, then Z_n is properly contained in a special subvariety. If the intersection $Z \cap T_m Z$ is proper, then we replace Z by an F-component containing Z_n. At this point we have to replace the Shimura variety S by the smallest one containing Z (so that Z is Hodge generic) and then replace the group by the adjoint. This requires quite a lot of tedious verifications. Once this is done, we repeat the process. If the inclusion has not occured earlier, then eventually $\dim(Z) = \dim(Z_n) + 1$ and the comparison of the degree of Z with the degree of the Galois orbit of Z_n proves the inclusion $Z \subset T_m Z$.

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