A note on the Hermite–Rankin constant
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A note on the Hermite–Rankin constant

par KAZUOMI SAWATANI, TAKAO WATANABE et KENJI OKUDA

To the memory of Anne-Marie Bergé

Abstract. We generalize Poor and Yuen’s inequality to the Hermite–Rankin constant $\gamma_{n,k}$ and the Bergé–Martinet constant $\gamma'_{n,k}$. Moreover, we determine explicit values of some low-dimensional Hermite–Rankin and Bergé–Martinet constants by applying Rankin’s inequality and some inequalities proven by Bergé and Martinet to explicit values of $\gamma'_{5}, \gamma'_{7}, \gamma_{4,2}$ and $\gamma_{n}$ ($n \leq 8$).

1. Introduction

In the recent paper [PY3], Poor and Yuen proved an inequality among the Hermite constant $\gamma_{n}$, the Bergé–Martinet constant $\gamma'_{n}$ and the constant $c_{n}$ defined from the dyadic trace. They also studied the condition of the equality $(\gamma_{n})^{2} = n/c_{n}$ and applied this to determine low dimensional Bergé–Martinet’s constants $\gamma'_{5}, \gamma'_{6}$ and $\gamma'_{7}$.

In the first half of this paper, we generalize Poor and Yuen’s inequality to the Hermite–Rankin constant $\gamma_{n,k}$ and the Bergé–Martinet constant $\gamma'_{n,k}$. In the second half, we show by using Rankin’s inequality and some inequalities proven by Bergé and Martinet in [BM] that explicit values of $\gamma'_{5}, \gamma'_{7}, \gamma_{4,2}$ and $\gamma_{n}$ ($n \leq 8$) lead us to explicit values of $\gamma_{6,2}, \gamma'_{6,2}, \gamma_{8,2}, \gamma'_{8,2}, \gamma_{8,3}, \gamma'_{8,3}$ and $\gamma_{8,4} = \gamma'_{8,4}$.

2. A generalization of Poor and Yuen’s inequality

In order to define $\gamma_{n,k}$ and $\gamma'_{n,k}$, we start from the definition of type one functions. Let $V_{n}$ be the real vector space of $n \times n$ real symmetric matrices
and \( P_n \) the open cone of positive definite symmetric matrices in \( V_n \). We define the inner product \( \langle \cdot, \cdot \rangle : V_n \times V_n \rightarrow \mathbb{R} \) by
\[
\langle s, t \rangle = \text{Tr}(st)
\]
for \( s, t \in V_n \). A function \( \phi : P_n \rightarrow \mathbb{R}_{>0} \) is called a type one function if \( \phi \) satisfies
\[
\text{(i) } \phi(\lambda s) = \lambda \phi(s) \text{ for all } \lambda \in \mathbb{R}_{>0}, \ s \in P_n, \ \text{and}
\]
\[
\text{(ii) } \phi(s + t) \geq \phi(s) + \phi(t) \text{ for all } s, t \in P_n.
\]
In addition, if \( \phi \) is class invariant, i.e., \( \phi(g^tg) = \phi(s) \) holds for all \( s \in P_n \) and \( g \in GL_n(\mathbb{Z}) \), then \( \phi \) is called a type one class function. A type one function is continuous on \( P_n \) ([PY, Proposition 2.2]).

Let \( \phi \) be a type one function. The dual function \( \hat{\phi} : P_n \rightarrow \mathbb{R}_{>0} \) of \( \phi \) is defined to be
\[
\hat{\phi}(s) = \inf_{t \in P_n} \langle s, t \rangle / \phi(t).
\]
This \( \hat{\phi} \) is also a type one function, and the dual of \( \hat{\phi} \) equals \( \phi \). If \( \phi \) is class invariant, then so is \( \hat{\phi} \). A typical example of type one class functions is given by \( s \mapsto \sqrt{n}(\det s)^{1/n} \), which is self-dual.

In order to give another example of type one class functions, we fix a positive integer \( k \) with \( 1 \leq k \leq n - 1 \). Let \( \overline{P}_n \) be the closure of \( P_n \) in \( V_n \), i.e., \( \overline{P}_n \) is the closed cone of positive semi-definite matrices. Define the function \( m_k : \overline{P}_n \rightarrow \mathbb{R}_{\geq 0} \) by
\[
m_k(s) = \inf_{x_1, \ldots, x_k \in \mathbb{Z}^n \atop x_1 \wedge \cdots \wedge x_k \neq 0} \det((t^{i}x_isx_j)_{i,j})
\]
for \( s \in \overline{P}_n \). It is obvious that \( m_k(s) > 0 \) if \( s \in P_n \), and \( m_k(s) = 0 \) otherwise.

**Lemma 2.1.** The function \( m'_k(s) = m_k(s)^{1/k} \) in \( s \in P_n \) is a type one class function.

**Proof.** The class invariance and the condition (i) for \( m'_k \) are trivial. We show the condition (ii). Since the function \( x \mapsto (\det x)^{1/k} \) in \( x \in P_k \) is a type one function, we have
\[
m'_k(s + t) = \inf_{x_1, \ldots, x_k \in \mathbb{Z}^n \atop x_1 \wedge \cdots \wedge x_k \neq 0} \det((t^{i}x_i(s + t)x_j)_{i,j})^{1/k}
\]
\[
\geq \inf_{x_1, \ldots, x_k \in \mathbb{Z}^n \atop x_1 \wedge \cdots \wedge x_k \neq 0} \left\{ \det((t^{i}x_isx_j)_{i,j})^{1/k} + \det((t^{i}x_itx_j)_{i,j})^{1/k} \right\}
\]
\[
\geq \inf_{x_1, \ldots, x_k \in \mathbb{Z}^n \atop x_1 \wedge \cdots \wedge x_k \neq 0} \det((t^{i}x_isx_j)_{i,j})^{1/k} + \inf_{x_1, \ldots, x_k \in \mathbb{Z}^n \atop x_1 \wedge \cdots \wedge x_k \neq 0} \det((t^{i}x_itx_j)_{i,j})^{1/k}
\]
\[
= m'_k(s) + m'_k(t).
\]
This shows that \( m'_k \) is a type one function. \( \square \)
If $k = 1$, then $m(s) = m'_1(s)$ is none other than the minimum of the quadratic form $x \mapsto txs$ on the set $\mathbb{Z}^n \setminus \{0\}$ of non-zero lattice points. The dual of $m$ is the dyadic trace, i.e.,

$$w(s) = \hat{m}(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle}{m(t)},$$

which plays an important role in Poor–Yuen’s theory.

By using $m_k$, the Hermite–Rankin constant $\gamma_{n,k}$ and the Bergé–Martinet constant $\gamma'_{n,k}$ are defined as follows:

$$\gamma_{n,k} = \sup_{s \in P_n} \frac{m_k(s)}{(\det s)^{k/n}} \quad \text{and} \quad \gamma'_{n,k} = \sup_{s \in P_n} \sqrt{m_k(s)m_k(s^{-1})}. $$

A generalization of $w(s)/m(s)$ is given by

$$c_{n,k}(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle^k}{m_k(s)m_k(t)}.$$

We set

$$c_{n,k} = \inf_{s \in P_n} c_{n,k}(s) = \inf_{(s, t) \in P_n \times P_n} \frac{\langle s, t \rangle^k}{m_k(s)m_k(t)}.$$

**Lemma 2.2.** The infimum

$$\inf_{(s, t) \in P_n \times P_n} \frac{\langle s, t \rangle^k}{m_k(s)m_k(t)}$$

is attained at some $(s_0, t_0) \in P_n \times P_n$.

**Proof.** Since $m_k(s) \leq \gamma_{n,k}(\det s)^{k/n}$ holds for all $s \in \overline{P}_n$, the function $s \mapsto m_k(s)$ is continuous on $\overline{P}_n$. Then, by [PY, Lemma 3.6], the infimum

$$w_k(s) = \inf_{t \in P_n} \frac{\langle s, t \rangle}{m_k(t)^{1/k}}$$

is attained at some $t_s \in P_n$. Since $w_k$ is the dual of $m'_k$, $w_k$ is also a type one class function, and hence $w_k(s)/m_k(s)^{1/k}$ depends only on the similar $GL_n(\mathbb{Z})$-equivalent class $R > 0 \cdot [s]$ of $s$. Therefore, we have

$$c_{n,k}^{1/k} = \inf_{s \in P_n} \frac{w_k(s)}{m_k(s)^{1/k}} = \inf_{[s] \in P_n/GL_n(\mathbb{Z})} \frac{w_k(s)}{m_k(s)=1}.$$

For a constant $c > c_{n,k}$, we consider the non-empty set

$$\Omega_c = \{ [s] \in P_n/GL_n(\mathbb{Z}) : m_k(s) = 1 \text{ and } w_k(s) \leq c^{1/k} \}.$$

From the inequality of the arithmetic and the geometric means, it follow that

$$(2.1) \quad n(\det s)^{1/n}(\det t)^{1/n} \leq \langle s, t \rangle$$
for all \((s, t) \in P_n \times P_n\) (cf. [BC, Theorem 1]). By using (2.1) and the definition of \(\gamma_{n,k}\), we have

\[
n \cdot \frac{(\det s)^{1/n}}{\gamma_{n,k}^{1/k}} \leq \frac{\langle s, t_s \rangle}{m_k(t_s)^{1/k}} = w_k(s).
\]

This implies that \(\Omega_c\) is a subset of

\[
\Omega'_c = \{ [s] \in P_n/GL_n(\mathbb{Z}) : m_k(s) = 1 \text{ and } \det s \leq n^{-n} c_{n/k}^{n/k} (\gamma_{n,k})^{n/k} \}.
\]

As a consequence, \(c_{n,k}^{1/k}\) is represented by

\[
c_{n,k}^{1/k} = \inf_{[s] \in \Omega'_c} w_k(s).
\]

Since \(\Omega'_c\) is compact in \(P_n/GL_n(\mathbb{Z})\) by [C, Proposition 2.2] and \(w_k\) is continuous, this infimum is attained at some \([s_0] \in \Omega'_c\). \(\square\)

In the case of \(k = 1\), Poor and Yuen proved the inequality:

\[
(2.2) \quad \frac{1}{(\gamma_{n,1})^2} \leq \frac{c_{n,1}}{n} \leq \frac{1}{(\gamma'_{n,1})^2}.
\]

We note that Barnes and Cohn ([BC]) also proved the first inequality of (2.2).

**Theorem 2.1.** For \(1 \leq k \leq n - 1\), we have

\[
\frac{1}{(\gamma_{n,k})^2} \leq \frac{c_{n,k}}{n^k} \leq \frac{1}{(\gamma'_{n,k})^2}.
\]

The equality \((\gamma_{n,k})^{-2} = c_{n,k}/n^k\) holds if and only if \(\gamma_{n,k} = \gamma'_{n,k}\).

**Proof.** By the inequality (2.1), we have

\[
n^k \cdot \inf_{s \in P_n} \frac{(\det s)^{k/n}}{m_k(s)} \cdot \inf_{t \in P_n} \frac{(\det t)^{k/n}}{m_k(t)} \leq c_{n,k},
\]

and hence

\[
n^k \left( \frac{1}{\gamma_{n,k}} \right)^2 \leq c_{n,k}.
\]

By the definition of \(c_{n,k}\), the inequality

\[
c_{n,k} \leq \frac{\langle s, s^{-1} \rangle^k}{m_k(s)m_k(s^{-1})} = \frac{n^k}{m_k(s)m_k(s^{-1})}
\]

holds for all \(s \in P_n\). Therefore,

\[
m_k(s)m_k(s^{-1}) \leq \frac{n^k}{c_{n,k}}.
\]
This gives
\[(\gamma_{n,k}')^2 \leq \frac{n^k}{c_{n,k}}.\]

It is known that the equality
\[n(\det s)^{1/n}(\det t)^{1/n} = \langle s, t \rangle\]
holds if and only if \(t\) is similar to \(s^{-1}\). By Lemma 2.2, there exists \((s_0, t_0) \in P_n \times P_n\) such that
\[c_{n,k} = \frac{\langle s_0, t_0 \rangle^k}{m_k(s_0)m_k(t_0)}.\]

In general, we have
\[\frac{n^k}{(\gamma_{n,k})^2} \leq \frac{n^k(\det s_0)^{k/n}(\det t_0)^{k/n}}{m_k(s_0)m_k(t_0)} \leq \frac{\langle s_0, t_0 \rangle^k}{m_k(s_0)m_k(t_0)} = c_{n,k}.\]

Thus, if \((\gamma_{n,k})^{-2} = c_{n,k}/n^k\) holds, then \(t_0\) must be similar to \(s_0^{-1}\) and \(\gamma_{n,k}\) is attained at both \(s_0\) and \(s_0^{-1}\). This implies \(\gamma_{n,k} = \gamma_{n,k}'.\)

\[\square\]

3. Explicit values of some Hermite–Rankin constants

In [BM], Bergé and Martinet proved several inequalities involving \(\gamma_{n,k}\) and \(\gamma_{n,k}'\), and determined the values \(\gamma_{2,1}' = 2/\sqrt{3}\), \(\gamma_{3,1}' = \sqrt{3/2}\) and \(\gamma_{4,1}' = \sqrt{2}\). In the inequality (2.2), Poor and Yuen proved the equality \((\gamma_{n,1}')^2 = n/c_{n,1}\) holds for \(n \leq 8\) and \(n = 24\) ([PY3, Theorem 2.4]). By using this, they determined the following explicit values of Bergé–Martinet constants:

\[\gamma_{5,1}' = \sqrt{2}, \quad \gamma_{6,1}' = \sqrt{8/3}, \quad \gamma_{7,1}' = \sqrt{3}.\]

These explicit values give the following:

**Theorem 3.1.** \(\gamma_{8,3} = \gamma_{8,4} = \gamma_{8,4}' = 4.\)

**Proof.** If \(n\) is even, then the equality \(\gamma_{n,n/2} = \gamma_{n,n/2}'\) holds in general ([M, Corollary 2.8.8]). We recall Rankin’s inequality:

\[(3.1) \quad \gamma_{n,k} \leq \gamma_{h,k}(\gamma_{n,h})^{k/h}\]

holds for \(1 \leq k < h < n\). In the case of \(n = 8, h = 4, k = 1\), we have

\[\left(\frac{\gamma_{8,1}}{\gamma_{4,1}}\right)^4 \leq \gamma_{8,4}.\]

Since \(\gamma_{8,1} = 2\) and \(\gamma_{4,1} = \sqrt{2}\), this gives \(4 \leq \gamma_{8,4}\). On the other hand, the following inequality is known for \(1 \leq k \leq n/2\) ([M, Theorem 2.8.7]):

\[\gamma_{n,2k}' \leq (\gamma_{n-k,k}')^2.\]
We use (3.2) twice. Namely, if we put \( n = 8, k = 2 \) and \( n = 6, k = 1 \), then
\[
4 \leq \gamma_{8,4} = \gamma'_{8,4} \leq (\gamma'_{6,2})^2 \leq (\gamma'_{5,1})^4 = 4.
\]
By applying (3.1) to the cases \( n = 8, h = 4, k = 3 \) and \( n = 8, h = 3, k = 1 \), we obtain
\[
\gamma_{8,3} \leq \gamma_{4,3}(\gamma_{8,4})^{3/4} = 4^{1/4} \cdot 4^{3/4} = 4
\]
and
\[
4 = \left(\frac{\gamma_{8,1}}{\gamma_{3,1}}\right)^3 \leq \gamma_{8,3}.
\]

**Corollary 3.1.** \( \gamma'_{6,2} = 2, \gamma_{6,2} = 3^{2/3} \).

**Proof.** By (3.3), \( \gamma'_{6,2} = 2 \) is trivial. Rankin’s inequality (3.1) gives for \( n = 6, h = 2, k = 1 \)
\[
\left(\frac{\gamma_{6,1}}{\gamma_{2,1}}\right)^2 \leq \gamma_{6,2}.
\]
Since, \( \gamma_{6,1} = (64/3)^{1/6} \) and \( \gamma_{2,1} = \sqrt{4/3} \), we have \( 3^{2/3} \leq \gamma_{6,2} \). On the other hand, the inequality
\[
(\gamma_{n,k})^n \leq (\gamma_{n-k,k})^{n-k}(\gamma'_{n,k})^{2k}
\]
is known for \( 1 \leq k \leq n/2 \) ([M, Theorem 2.8.7]). By putting \( n = 6, k = 2 \), we obtain
\[
(\gamma_{6,2})^6 \leq (\gamma_{4,2} \cdot \gamma'_{6,2})^4 = 3^4
\]
because of \( \gamma_{4,2} = 3/2 \). \( \square \)

**Corollary 3.2.** \( \gamma_{8,2} = \gamma'_{8,2} = 3 \).

**Proof.** Rankin’s inequality gives for \( n = 8, h = 2, k = 1 \)
\[
3 = \left(\frac{\gamma_{8,1}}{\gamma_{2,1}}\right)^2 \leq \gamma_{8,2}.
\]
By (3.2), we have
\[
\gamma'_{8,2} \leq (\gamma'_{7,1})^2 = 3.
\]
Then, by (3.4),
\[
3 \leq \gamma_{8,2} \leq (\gamma_{6,2})^{3/4}(\gamma'_{8,2})^{1/2} \leq 3.
\]
\( \square \)

We show that all \( \gamma_{8,2}, \gamma_{8,3} \) and \( \gamma_{8,4} \) are attained on the \( E_8 \) lattice and \( \gamma_{6,2} \) is attained on the \( E_6 \) lattice. For a full lattice \( \Lambda \) in the Euclidean space \( \mathbb{R}^n \), \( \det \Lambda \) denotes the determinant of \( \Lambda \), i.e., \( \det \Lambda = (\det g)^2 \) if \( \Lambda = g\mathbb{Z}^n \).
with $g \in GL_n(\mathbb{R})$. The Hermite–Rankin invariant $\gamma_{n,k}(\Lambda)$ and the Bergé–Martinet invarinat $\gamma'_{n,k}(\Lambda)$ of $\Lambda$ are defined as

$$\gamma_{n,k}(\Lambda) = \inf_{\Lambda'} \gamma(\Lambda, \Lambda'),$$

where $\gamma(\Lambda, \Lambda') = \frac{\det \Lambda'}{(\det \Lambda)^{k/n}}$ and $\Lambda'$ runs over all sublattices in $\Lambda$ of rank $k$, and then

$$\gamma'_{n,k}(\Lambda) = \sqrt{\gamma_{n,k}(\Lambda) \gamma_{n,k}(\Lambda^*)},$$

where $\Lambda^*$ denotes the dual lattice of $\Lambda$. In terms of lattices, $\gamma_{n,k}$ and $\gamma'_{n,k}$ are given by

$$\gamma_{n,k} = \sup_{\Lambda} \gamma_{n,k}(\Lambda), \quad \gamma'_{n,k} = \sup_{\Lambda} \gamma'_{n,k}(\Lambda),$$

where $\Lambda$ runs over all full lattices in $\mathbb{R}^n$.

**Proposition 3.1.** One has $\gamma_{8,2} = \gamma_{8,2}(E_8) = \gamma(E_8, A_2)$, $\gamma_{8,3} = \gamma_{8,3}(E_8) = \gamma(E_8, A_3)$, $\gamma_{8,4} = \gamma_{8,4}(E_8) = \gamma(E_8, D_4)$ and $\gamma_{6,2} = \gamma_{6,2}(E_6) = \gamma(E_6, A_2)$.

**Proof.** From the proof of Rankin’s inequality, it follows that

$$\gamma_{n,k}(\Lambda) \leq \gamma_{h,k} \cdot (\gamma_{n,h}(\Lambda))^{k/h}$$

holds for any lattice $\Lambda$ of rank $n$ and $1 \leq k < h < n$ (cf. [M, Proof of Theorem 2.8.6]). For $\Lambda = E_8$ and $n = 8, h = 2, k = 1$, (3.5) gives

$$2 = \gamma_{8,1}(E_8) \leq \frac{2}{\sqrt{3}} \cdot (\gamma_{8,2}(E_8))^{1/2}.$$ 

This implies

$$3 \leq \gamma_{8,2}(E_8) \leq 3.$$ 

From the table [M, Table 4.10.13], it follows $\gamma(E_8, A_2) = 3$. By a similar fashion, we have $\gamma_{8,3} = \gamma_{8,3}(E_8)$, $\gamma_{8,4} = \gamma_{8,4}(E_8)$ and $\gamma_{6,2} = \gamma_{6,2}(E_6)$.

We note that Coulangeon computed $\gamma_{8,2}(E_8) = 3, \gamma_{7,2}(E_7) = 3/2^{2/7}$ and $\gamma_{6,2}(E_6) = 3^{2/3}$ by another method ([C, Théorème 5.1.1]).

**Corollary 3.3.** $\gamma'_{8,3} = 4$.

**Proof.** Since $E_8$ is self-dual, the Bergé–Martinet invariant $\gamma'_{8,3}(E_8)$ coincides with $\gamma_{8,3}(E_8)$, i.e., $\gamma'_{8,3}(E_8) = 4$. Then, by [M, Proposition 2.8.4],

$$4 = \gamma'_{8,3}(E_8) \leq \gamma'_{8,3} \leq \gamma_{8,3} = 4.$$ 

By Theorem 3.1, Corollaries 3.2, 3.3 and the duality relation ([M, Proposition 2.8.5]), Hermite–Rankin and Bergé–Martinet constants of dimension 8 are completely determined.
Proposition 3.2. One has

\[ \frac{4}{\sqrt{3}} \leq \gamma_{6,3} \leq \sqrt{6} \quad \text{and} \quad 2^{11/7} \leq \gamma_{7,3} \leq 2^{4/7} \cdot 3^{2/3}. \]

These lower bounds are attained on \( E_6 \) and \( E_7 \), i.e., we have \( \gamma_{6,3}(E_6) = \gamma(E_6, A_3) = \frac{4}{\sqrt{3}} \), \( \gamma_{7,3}(E_7) = \gamma(E_7, A_3) = 2^{11/7} \) and moreover

\[ \gamma'_{7,2}(E_7) = \sqrt{6}, \quad \gamma'_{7,3}(E_7) = 2\sqrt{2}. \]

**Proof.** From Rankin’s inequality (3.1) for \( n = 8, h = 6, k = 3 \), it follows \( 4/\sqrt{3} \leq \gamma_{6,3} \). All other inequalities of (3.6) are also obtained by Rankin’s inequality. Let \( \Lambda' \) be a sublattice of \( E_6 \) of rank 3 which attains \( \gamma_{6,3}(E_6) \). Since \( \gamma(E_8, E_6) = 3 \), we have

\[ \gamma(E_8, \Lambda') = \det \Lambda' = \gamma_{6,3}(E_6)\gamma(E_8, E_6)^{1/2}. \]

Therefore,

\[ 4 = \gamma_{8,3}(E_8) \leq \gamma(E_8, \Lambda') = \gamma_{6,3}(E_6)\sqrt{3}. \]

We know \( \gamma(E_6, A_3) = 4/\sqrt{3} \) by [M, Table 4.10.13], and hence

\[ \frac{4}{\sqrt{3}} \leq \gamma_{6,3}(E_6) \leq \gamma(E_6, A_3) = \frac{4}{\sqrt{3}}. \]

Similarly, we obtain \( \gamma_{7,3}(E_7) = 2^{11/7}, \gamma_{7,4}(E_7) = \gamma(E_7, D_4) = 2^{10/7} \), and \( \gamma_{7,5}(E_7) = \gamma(E_7, D_5) = 2^{9/7} \). Then \( \gamma'_{7,2}(E_7) = \sqrt{\gamma_{7,2}(E_7)\gamma_{7,5}(E_7)} = \sqrt{6} \) and \( \gamma'_{7,3}(E_7) = \sqrt{\gamma_{7,3}(E_7)\gamma_{7,4}(E_7)} = 2\sqrt{2} \) follows from the duality relation \( \gamma_{n,k}(\Lambda^*) = \gamma_{n,n-k}(\Lambda) \) ([M, Proposition 2.8.5]). \( \square \)

The same argument as in the proof of Proposition 3.2 yields \( \gamma_{6,4}(E_6) = \gamma_{6,2}(E_6) = 4/3^{2/3} \). This implies \( \gamma'_{6,2}(E_6) = 2 = \gamma'_{6,2} \), namely \( \gamma'_{6,2} \) is attained on \( E_6 \). Since \( E_8 \) is self-dual, all \( \gamma'_{8,2}, \gamma'_{8,3}, \gamma'_{8,4} \) are attained on \( E_8 \).

Mayer [Ma, Théorème 3.59] extended Theorem 2.1 to generalized Hermite constants over an algebraic number field and applied it to determine the values \( \gamma'_{3,1}(Q(\sqrt{-1})) = \gamma'_{3,1}(Q(\sqrt{-3})) = 2, \gamma'_{4,1}(Q(\sqrt{-1})) = 4, \gamma'_{4,1}(Q(\sqrt{-3})) = 3 \) and \( \gamma_{4,2}(Q(\sqrt{-1})) = \gamma_{4,2}(Q(\sqrt{-3})) = 4 \) ([Ma, Propositions 3.66, 3.67]).

**References**


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