JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 22, n° 3 (2010), p. 727-740.

<http://jtnb.cedram.org/item?id=JTNB_2010__22_3_727_0>

Journal de Théorie des Nombres de Bordeaux **22** (2010), 727-740

A generalization of Voronoï's Theorem to algebraic lattices

par Kenji OKUDA et Syouji YANO

Résumé. Soient K un corps de nombres et $\mathcal{O}_K$ l'anneau des entiers de K. Dans cet article, nous prouvons un analogue du théorème de Voronoï pour les $\mathcal{O}_K$-réseaux, et la finitude du nombre de classes de $\mathcal{O}_K$-réseaux parfaits, à similitude près.

Abstract. Let K be an algebraic number field and $\mathcal{O}_K$ the ring of integers of K. In this paper, we prove an analogue of Voronoï's theorem for $\mathcal{O}_K$-lattices and the finiteness of the number of similar isometry classes of perfect $\mathcal{O}_K$-lattices.

1. Introduction

Let K be an algebraic number field of degree n and $\mathcal{O}_K$ the ring of integers of K. The purpose of this paper is to generalize Voronoï's theorem to the Hermite function defined on the set of $\mathcal{O}_K$-lattices.

Let $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$ and $K_{\mathbb{R}}^m = K^m \otimes_{\mathbb{Q}} \mathbb{R}$. An element of $K_{\mathbb{R}}^m$ is denoted as a column vector with entries in $K_{\mathbb{R}}$. As an $\mathbb{R}$-vector space, $K_{\mathbb{R}}^m$ is equipped with the inner product $< \, , \, >$ defined by

$$< x, y >= \mathrm{Tr}_{K_{\mathbb{R}}}({}^t\overline{x}y)$$

for $x, y \in K_{\mathbb{R}}^m$, where $\mathrm{Tr}_{K_{\mathbb{R}}}$ stands for the trace of the étale algebra $K_{\mathbb{R}}$ over $\mathbb{R}$ and $\overline{x}$ stands for the conjugate of x. An $\mathcal{O}_K$-submodule Λ in $K_{\mathbb{R}}^m$ is called an $\mathcal{O}_K$-lattice if Λ is discrete and $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = K_{\mathbb{R}}^m$. The set of all $\mathcal{O}_K$-lattices in $K_{\mathbb{R}}^m$ is denoted by $\mathcal{L}$. For $\Lambda \in \mathcal{L}$, the minimum $Q(\Lambda)$ and the determinant $\det(\Lambda)$ of Λ are defined by

$$Q(\Lambda) = \min_{x \in \Lambda \setminus \{0\}} < x, x > \quad \text{and} \quad \det(\Lambda) = \left(\frac{\omega(K_{\mathbb{R}}^m/\Lambda)}{\omega(K_{\mathbb{R}}^m/\mathcal{O}_K^m)} \right)^2,$$

where ω denotes an invariant measure on $K_{\mathbb{R}}^m$. The Hermite function $\gamma_K^{\dagger} : \mathcal{L} \longrightarrow \mathbb{R}_{>0}$ is defined to be

$$\gamma_K^{\dagger}(\Lambda) = \frac{Q(\Lambda)}{\det(\Lambda)^{1/mn}}.$$

Manuscrit reçu le 19 janvier 2010.

An $\mathcal{O}_K$-lattice $\Lambda \in \mathcal{L}$ is said to be extreme if $\gamma_K^\dagger$ attains a local maximum on Λ up to multiplication by an element of $\mathbb{R}^\times$. In the case of $K = \mathbb{Q}$, Voronoï's theorem says that Λ is extreme if and only if Λ is perfect and eutactic. We extend this theorem to general K. Leibak ([6], Theorem 5) gave a weak version of Voronoï's theorem for K. However Leibak's definition of eutaxy is insufficient to show the extremeness. We modify Leibak's definition of eutaxy and prove the complete analogue of Voronoï's theorem. At the end of this paper we prove that the number of similar isometry classes of perfect $\mathcal{O}_K$-lattices in $K_\mathbb{R}^m$ is finite.

2. $\mathcal{O}_K$-lattices

Let K be an algebraic number field of degree n and $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R}$ an étale algebra over $\mathbb{R}$. Assume K has r real embeddings $\sigma_1, \cdots, \sigma_r$ and $2s$ complex embeddings $\sigma_{r+1}, \cdots, \sigma_{r+2s}$, where $r+2s = n$ and σ_{r+s+i} is the composition of the complex conjugate "$^-$" and σ_{r+i} for $1 \le i \le s$. Then $K_\mathbb{R}$ is identified with $\mathbb{R}^r \times \mathbb{C}^s$ and K is included in $K_\mathbb{R}$ by $x \longrightarrow (\sigma_1(x), \cdots, \sigma_{r+s}(x))$. The trace of $K_\mathbb{R}$ over $\mathbb{R}$ is defined by

$$\mathrm{Tr}_{K_\mathbb{R}}(x) = \sum_{i=1}^{r} x_i + \sum_{i=1}^{s} (x_{r+i} + \overline{x_{r+i}})$$

for $x = (x_1, \cdots, x_{r+s}) \in K_\mathbb{R}$, where $\overline{x_{r+i}}$ is the complex conjugate of x_{r+i}. For $x = (x_1, \cdots, x_{r+s}) \in K_\mathbb{R}$ we denote $\overline{x} = (x_1, \cdots, x_r, \overline{x_{r+1}}, \cdots, \overline{x_{r+s}})$.

Let $K_\mathbb{R}^m = K^m \otimes_\mathbb{Q} \mathbb{R}$ and $<\ ,\ >$ the inner product of $K_\mathbb{R}^m$ which is defined in Section 1. For $x \in K_\mathbb{R}^m$, we set

$$Q(x) =< x, x > \qquad \text{and} \qquad ||x|| = Q(x)^{1/2}.$$

The group of $K_\mathbb{R}$-linear automorphisms of $K_\mathbb{R}^m$ is denoted by $GL_m(K_\mathbb{R})$, which is identified with $GL_m(\mathbb{R})^r \times GL_m(\mathbb{C})^s$. The group of isometries with respect to Q is denoted by $O_m(K_\mathbb{R})$, i.e.

$$O_m(K_\mathbb{R}) = \{g \in GL_m(K_\mathbb{R}) \mid \ < gx, gy >=< x, y > \text{ for all } x, y \in K_\mathbb{R}^m\}.$$

We denote the set of self-adjoint matrices by

$$H_m(K_\mathbb{R}) = \{V \in M_m(K_\mathbb{R}) \mid \ < Vx, y >=< x, Vy > \text{ for all } x, y \in K_\mathbb{R}^m\}.$$

According to the identification $K_\mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s$, the set $H_m(K_\mathbb{R})$ is identified with $Sym_m(\mathbb{R})^r \times H_m(\mathbb{C})^s$, where $Sym_m(\mathbb{R})$ (resp. $H_m(\mathbb{C})$) denote the set of $m \times m$ real symmetric (resp. Hermitian) matrices. The trace TR on $H_m(K_\mathbb{R})$ is defined to be

$$\mathrm{TR}(V) = \mathrm{Tr}_{K_\mathbb{R}}((\mathrm{Tr}(V_1), \cdots, \mathrm{Tr}(V_{r+s})))$$

for $V = (V_1, \cdots, V_{r+s})$ in $H_m(K_\mathbb{R})$. The dual space of $H_m(K_\mathbb{R})$ as an $\mathbb{R}$ vector space is denoted by $H_m(K_\mathbb{R})^*$. A self-adjoint matrix V is said to be positive definite (resp. semi-positive definite) if $< Vx, x >> 0$ (resp.

$< Vx, x > \geq 0$) for all $x \in K_{\mathbb{R}}^m \backslash \{0\}$. We denote the set of positive definite (resp. semi-positive definite) self-adjoint matrices in $H_m(K_{\mathbb{R}})$ by $H_m^{++}(K_{\mathbb{R}})$ (resp. $H_m^{+}(K_{\mathbb{R}})$.)

For any $\mathcal{O}_K$-lattice $\Lambda \subset K_{\mathbb{R}}^m$, there exists some $g \in GL_m(K_{\mathbb{R}})$ such that $g^{-1}\Lambda$ is a projective $\mathcal{O}_K$-module in K^m (cf. [5], Lemma 3.2). By Steinitz's theorem, any projective $\mathcal{O}_K$-module in K^m is isomorphic to $\mathcal{O}_K^{m-1} \oplus \mathfrak{A}$ for some ideal $\mathfrak{A}$ in $\mathcal{O}_K$. Let $\mathfrak{A}_1 = \mathcal{O}_K, \mathfrak{A}_2, \cdots, \mathfrak{A}_h$ be a complete system of representatives of the ideal class group of K. Let $\Lambda_i = \mathcal{O}_K^{m-1} \oplus \mathfrak{A}_i$ for $1 \leq i \leq h$. Then the set of all $\mathcal{O}_K$-lattices of $K_{\mathbb{R}}^m$ is given by the disjoint union

$$\mathcal{L} = \coprod_{i=1}^{h} \mathcal{L}_i,$$

where $\mathcal{L}_i$ is the $GL_m(K_{\mathbb{R}})$-orbit of Λ_i. Each $\mathcal{L}_i$ is identified with $GL_m(K_{\mathbb{R}})/GL(\Lambda_i)$, where $GL(\Lambda_i)$ denotes the stabilizer of Λ_i in $GL_m(K_{\mathbb{R}})$.

Let $\Lambda \subset K_{\mathbb{R}}^m$ be an $\mathcal{O}_K$-lattice. We denote the set of shortest vectors in Λ by

$$S(\Lambda) = \{x \in \Lambda \mid Q(x) = Q(\Lambda)\}.$$

Definition. For $x \in K_{\mathbb{R}}^m$, we define the $\mathbb{R}$-linear form $\varphi_x \in H_m(K_{\mathbb{R}})^*$ by

$$\varphi_x(V) = < Vx, x > \text{ for } V \in H_m(K_{\mathbb{R}}).$$

Definition. Let $\Lambda \subset K_{\mathbb{R}}^m$ be an $\mathcal{O}_K$-lattice.

(1) Λ is said to be perfect if $\{\varphi_x | x \in S(\Lambda)\}$ generates $H_m(K_{\mathbb{R}})^*$.
(2) Λ is said to be eutactic if there exist $\rho_x \in \mathbb{R}_{>0}$ for all $x \in S(\Lambda)$ such that $\mathrm{TR} = \sum_{x \in S(\Lambda)} \rho_x \varphi_x$.
(3) Λ is said to be extreme if the function $\gamma_K^{\dagger}$ attains a local maximum on Λ.

This definition of perfection is same as the definition of perfection that appeared in ([3], Section 3).

3. Some rationality of perfect $\mathcal{O}_K$-lattices

In this section we prove some rationality of perfect $\mathcal{O}_K$-lattices. We call two $\mathcal{O}_K$-lattices Λ and Λ' are isometry if $\Lambda = T\Lambda'$ for some $T \in O_m(K_{\mathbb{R}})$. For $A = (a_{ij}) \in M_m(K_{\mathbb{R}})$, we put the $A^* = {}^t(\overline{a_{ij}})$.

Theorem 3.1. *A perfect $\mathcal{O}_K$-lattice Λ in $K_{\mathbb{R}}^m$ is determined up to isometry by the set of minimum vectors $S(\Lambda)$ and the minimum $Q(\Lambda)$.*

Proof. Let $N = rm(m+1)/2 + sm^2$. By the argument in Section 2, an $\mathcal{O}_K$-lattice Λ is denoted by $\Lambda = g\Lambda_0$ for some $g \in GL_m(K_{\mathbb{R}})$ and some projective $\mathcal{O}_K$-module $\Lambda_0 \subset K^m$. Let $H = g^*g \in H_m^{++}(K_{\mathbb{R}})$, and $S(H)$

denotes the subset $g^{-1}S(\Lambda)$ of Λ_0. Then $\{\varphi_{x_0}\}_{x_0 \in S(H)}$ also spans $H_m(K_{\mathbb{R}})^*$ since Λ is perfect. We have

$$\varphi_x(V) =< Vx, x >=< g^*Vgx_0, x_0 >= \varphi_{x_0}(g^*Vg)$$

for any $x = gx_0 \in S(\Lambda)$ and $V \in H_m(K_{\mathbb{R}})$. We consider a system of linear equations in N variables which consist of matrix elements of $V' \in H_m(K_{\mathbb{R}})$ such that

$$\varphi_{x_0}(V') = Q(\Lambda) \qquad (x_0 \in S(H)).$$

The coefficient of these linear equations are contained in the Galois closure of K. By the perfection of Λ, H is a unique solution of this system of linear equations. Therefore an isometry class of perfect $\mathcal{O}_K$-lattice is uniquely determined by the set of minimum vectors and the minimum. □

The following is obvious by Cramer's formula.

Corollary 3.1. *If $\Lambda = g\Lambda_0$ is a perfect $\mathcal{O}_K$-lattice and $Q(\Lambda) = 1$, then all entries of g^*g are contained in the Galois closure of K over $\mathbb{Q}$.*

If K is a totally real or a CM-field (i.e. a totally imaginary quadratic extension over a totally real algebraic number field), then we have a stronger result. If K is a totally real field, we denote the set of $m \times m$ symmetric matrices with entries in K by $H_m(K)$, i.e., $H_m(K) = H_m(K_{\mathbb{R}}) \cap M_m(K)$. If K is a CM-field, then there exists a non-trivial involution ρ on K such that $\sigma_{r+1} \circ \rho = \sigma_{r+2}$. Hence we can define the set of $m \times m$ Hermitian matrices with entries in K by

$$H_m(K) = \{X \in M_m(K) \mid \rho({}^tX) = X\} = H_m(K_{\mathbb{R}}) \cap M_m(K).$$

In these cases, we have the following rationality of perfect $\mathcal{O}_K$-lattice.

Theorem 3.2. *Let K be a totally real or a CM-field. If $\mathcal{O}_K$-lattice $\Lambda = g\Lambda_0$ in $K_{\mathbb{R}}^m$ is perfect with $Q(\Lambda) = 1$, then $g^*g \in H_m(K)$.*

Proof. Let $N = rm(m+1)/2 + sm^2$. By the same argument as the proof of Theorem 3.1, there exist $x_i = gy_i \in S(\Lambda), i = 1, \cdots, N$ such that $\{\varphi_{x_i}\}_{i=1}^N$ spans $H_m(K_{\mathbb{R}})^*$. Then $\{\varphi_{y_i}\}_{i=1}^N$ also spans the dual space of $H_m(K)$ as a $\mathbb{Q}$ vector space.

Let $B_1, \cdots, B_N \in H_m(K)$ be the dual basis of $\varphi_{y_1}, \cdots, \varphi_{y_N}$ as a $\mathbb{Q}$ vector space. Then $B = \sum_{i=1}^N B_i$ satisfies $\varphi_{y_i}(B) = \varphi_{y_i}(B_i) = 1$ and

$$\varphi_{y_i}(g^*g) =< g^*gy_i, y_i >=< gy_i, gy_i >= Q(x_i) = 1.$$

Since the system of linear equations $\{\varphi_{y_i}(V) = 1\}_{i=1}^N$ have exactly one solution, $B = g^*g \in H_m(K)$. □

4. Main Theorem

In this section, we prove

Theorem 4.1. *Let $\Lambda \in \mathcal{L}$ be an $\mathcal{O}_K$-module. Then Λ is extreme if and only if Λ is perfect and eutactic.*

We prepare several Lemmas. The following Lemma 4.1 is proved in ([7], Theorem 3.3.1).

Lemma 4.1 (Stiemke)**.** *Let V be a vector space and $\varphi_1, \cdots, \varphi_k$ linear forms from V to $\mathbb{R}$. The following conditions are equivalent.*

(1) $\{x \in V | \varphi_i(x) \geq 0$ *for all* $i\} = \cap_{i=1}^{k} \mathrm{Ker}\varphi_i$.

(2) *There exist* $\rho_i \in \mathbb{R}_{>0}$ *such that* $\sum_{i=1}^{k} \rho_i \varphi_i = 0$.

The next is proved similarly as in ([7], Theorem 3.1.7 and Theorem 3.1.8).

Lemma 4.2. (1) *For any $V \in H_m^+(K_\mathbb{R})$, there uniquely exists $\sqrt{V} \in H_m^+(K_\mathbb{R})$ such that*

$$V = \sqrt{V}^{\,*}\sqrt{V} = \sqrt{V}\sqrt{V}.$$

(2) *For any $U \in GL_m(K_\mathbb{R})$, there uniquely exist $V', V'' \in H_m^{++}(K_\mathbb{R})$ and $O', O'' \in O_m(K_\mathbb{R})$ such that*

$$U = O'V' = V''O''.$$

As usual, the operator norm of $A \in M_m(K_\mathbb{R})$ is defined to be

$$||A|| = \sup_{x \in K_\mathbb{R}^m \setminus \{0\}} \frac{||Ax||}{||x||}.$$

Lemma 4.3. *There exists a neighborhood $\mathcal{U}$ of* Id *in $GL_m(K_\mathbb{R})$ such that*

$$S(U\Lambda) \subset U(S(\Lambda))$$

for all $U \in \mathcal{U}$.

Proof. Let $Q_1 = Q(\Lambda)$ and $Q_2 = \min_{x \in \Lambda, Q(x) > Q(\Lambda)} Q(x)$. Then the neighborhood of Id defined by

$$\mathcal{U} = \left\{ U \in GL_m(K_\mathbb{R}) \,\middle|\, ||U^{-1}|| < \sqrt{\frac{2Q_2}{Q_1 + Q_2}}, ||U|| < \sqrt{\frac{Q_1 + Q_2}{2Q_1}} \right\}$$

satisfies the claim, because we have

$$||Uy|| > \sqrt{\frac{Q_1 + Q_2}{2Q_2}} ||y|| \geq \sqrt{\frac{Q_1 + Q_2}{2}} = \sqrt{\frac{Q_1 + Q_2}{2Q_1}} ||x|| > ||Ux||$$

for all $x \in S(\Lambda)$ and all $y \in \Lambda \setminus (S(\Lambda) \cup \{0\})$. $\square$

Lemma 4.4. *There exists a neighborhood $\mathcal{V}$ of* 0 *in* $H_m(K_{\mathbb{R}})$ *such that* $\mathrm{Id}+V \in H_m^+(K_{\mathbb{R}})$ *for any* $V \in \mathcal{V}$ *and*

$$Q(\Lambda) = Q(\sqrt{\mathrm{Id}+V}\Lambda) \iff \min_{x \in S(\Lambda)} \varphi_x(V) = 0.$$

Proof. Let $\mathcal{U} \subset GL_m(K_{\mathbb{R}})$ be a neighborhood of Id such that $\mathcal{U}$ satisfies Lemma 4.3. Let $\mathcal{V} \subset H_m(K_{\mathbb{R}})$ be a neighborhood of 0 such that $\mathrm{Id}+V \in H_m^+(K_{\mathbb{R}})$ and $U = \sqrt{\mathrm{Id}+V}$ is contained in $\mathcal{U}$ for any $V \in \mathcal{V}$. Then we have $Q(U\Lambda) = \min_{x \in S(\Lambda)} Q(Ux)$ by Lemma 4.3, and that

$$\begin{aligned} Q(Ux) &= < Ux, Ux > = < U^*Ux, x > \\ &= \varphi_x(U^2) = \varphi_x(\mathrm{Id}+V) = Q(x) + \varphi_x(V). \end{aligned}$$

Therefore we have $Q(\Lambda) = Q(U\Lambda) \iff \min_{x \in S(\Lambda)} \varphi_x(V) = 0$. □

Lemma 4.5. (1) *There exists a neighborhood $\mathcal{V}$ of* 0 *in* $H_m(K_{\mathbb{R}})$ *such that $V \in \mathcal{V}$ satisfies $\sqrt{\mathrm{Id}+V} \in O_m(K_{\mathbb{R}})$ or* $\det(\sqrt{\mathrm{Id}+V}) < 1$ *if* $\mathrm{TR}(V) \leq 0$.

(2) *Let $\mathcal{C} \subset H_m(K_{\mathbb{R}})$ be a closed cone such that any $V \in \mathcal{C}\backslash\{0\}$ satisfies* $\mathrm{TR}(V) > 0$. *Then there exists $\alpha > 0$ such that*

$$V \in \mathcal{C} \text{ and } 0 < ||V|| < \alpha \Longrightarrow \det(\mathrm{Id}+V) > 1.$$

Proof. (1) Let $\{\lambda_{1,1}, \cdots, \lambda_{(r+s),m}\}$ be the set of eigenvalues of $V = (V_1, \cdots, V_{r+s}) \in \mathcal{V}$, where $\lambda_{i,j}$ is an eigenvalue of V_i. Then eigenvalues of $\mathrm{Id}+V$ are given by $1+\lambda_{1,1}, \cdots, 1+\lambda_{(r+s),m}$. Here we may assume that all of eigenvalues of $\mathrm{Id}+V$ are positive for any $V \in \mathcal{V}$ by taking $\mathcal{V}$ enough small. Hence two functions $\Psi_V(t) = \det(\mathrm{Id}+tV)$ and $\psi_V(t) = \log \Psi_V(t)$ are well defined for $t \in [0,1]$ and for $V \in \mathcal{V}$. Differentials of ψ_V are given by

$$\psi_V'(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\lambda_{i,j}}{1+\lambda_{i,j}t} \quad \text{and} \quad \psi_V''(t) = -\sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\lambda_{i,j}^2}{(1+\lambda_{i,j}t)^2},$$

where $\lambda_{(r+s+i),j} = \lambda_{(r+i),j}$. If $V = 0$, then it is obvious that $\sqrt{\mathrm{Id}+V} \in O_m(K_{\mathbb{R}})$. Hence we assume that $V \neq 0$ from here. Then we have $\psi_V''(t) < 0$ for all $t \in [0,1]$ and $\psi_V(0) = 0$. Under the condition that $\mathrm{TR}(V) \leq 0$, we have

$$\psi_V'(0) = \sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_{i,j} = \mathrm{TR}(V) \leq 0.$$

Consequently we have $\psi_V(t) \leq 0$ for $t \in [0,1]$, especially $\psi_V(1) < 0$. Therefore $\det(\sqrt{\mathrm{Id}+V}) = (\Psi_V(1))^{1/2} < 1$.
(2) Let $\Sigma = \{W \in H_m(K_{\mathbb{R}}) \mid ||W|| \leq 1\}$. We can define two functions $\Psi_W(t)$ and $\psi_W(t)$ as the proof of 1) for any $W \in \mathcal{C} \cap \Sigma$ and for enough small $t \geq 0$.

By the condition $\mathrm{TR}(W) > 0$, we have $\psi_W(0) = 0$ and $\psi_W'(0) > 0$. Hence there exists $t_W > 0$ such that $\psi_W(t) > 0$ for all $t \in [0, t_W]$. Since the function $\mathcal{C} \cap \Sigma \ni W' \longrightarrow \psi_{W'}(t_W) \in \mathbb{R}$ is continuous, there exists a open neighborhood $\mathcal{V}(W) \subset H_m(K_\mathbb{R})$ of W such that $\psi_{W'}(t_W) > 0$ for any $W' \in \mathcal{V}(W) \cap (\mathcal{C} \cap \Sigma)$. On the other hand, we have $\psi_{W'}(t) > 0$ for all $t \in (0, t_W]$ by the convexity of $\psi_{W'}(t)$.

Since $\mathcal{C} \cap \Sigma \subset \cup_{W \in \mathcal{C} \cap \Sigma} \mathcal{V}(W)$ is compact, there exist $W_1, \cdots, W_k \in \mathcal{C} \cap \Sigma$ such that $\cup_{i=1}^k \mathcal{V}(W_i) \supset \mathcal{C} \cap \Sigma$. Let $\alpha = \min(t_{W_1}, \cdots, t_{W_k})$. Let $V \in \mathcal{C}$ such that V satisfies the condition $0 < ||V|| < \alpha$ in the claim. Since $W = \frac{V}{||V||}$ is contained in $\mathcal{C} \cap \Sigma$, we have $\psi_W(t) > 0$ for all $t \in (0, \alpha)$. Then

$$\psi_W(||V||) = \sum_{i=1}^{n} \sum_{j=1}^{m} (1 + \frac{\lambda_{ij}}{||V||} ||V||) = \psi_V(1) > 0.$$

Therefore we have $\det(\mathrm{Id} + V) > 1$. $\square$

Proposition 4.1. *Let $\Lambda \in \mathcal{L}$ be an $\mathcal{O}_K$-module. The following conditions are equivalent.*

(1) *Λ is extreme.*
(2) *If $V \in H_m(K_\mathbb{R})$ satisfies $\min_{x \in S(\Lambda)} \varphi_x(V) = 0$ and $\mathrm{TR}(V) \leq 0$, then $V = 0$.*

Proof. ($1 \Rightarrow 2$) Let $\mathcal{V} \subset H_m(K_\mathbb{R})$ be a neighborhood of 0 which satisfies Lemma 4.4 and Lemma 4.5. Let $V \in H_m(K_\mathbb{R})$ such that V satisfies $\min_{x \in S(\Lambda)} \varphi_x(V) = 0$ and $\mathrm{TR}(V) \leq 0$. Since Λ is extreme, $\gamma_K^\dagger$ attains a local maximum on Λ in some neighborhood $\mathcal{V}_\Lambda$ of Λ. For enough small $\lambda > 0$, λV is contained in $\mathcal{V}$ and then $\sqrt{\mathrm{Id} + \lambda V}(\Lambda)$ is in $\mathcal{V}_\Lambda$. From the equivalence proved in Lemma 4.4, we have $Q(\Lambda) = Q(\sqrt{\mathrm{Id} + \lambda V}(\Lambda))$. Moreover we have $\det\sqrt{\mathrm{Id} + \lambda V} < 1$ or $V = 0$ by Lemma 4.5. If $\det\sqrt{\mathrm{Id} + \lambda V} < 1$, then it is contradict to the extremeness of Λ since

$$\gamma_K^\dagger(\sqrt{\mathrm{Id} + \lambda V}(\Lambda)) = \frac{Q(\sqrt{\mathrm{Id} + \lambda V}(\Lambda))}{\det(\sqrt{\mathrm{Id} + \lambda V}(\Lambda))^{1/nm}} > \frac{Q(\Lambda)}{\det(\Lambda)^{1/nm}} = \gamma_K^\dagger(\Lambda).$$

Therefore we have $V = 0$.
($1 \Leftarrow 2$) We may assume that $Q(\Lambda) = 1$. Let

$$\mathcal{C} = \{V \in H_m(K_\mathbb{R}) \mid \min_{x \in S(\Lambda)} \varphi_x(V) \geq 0\}.$$

In order to apply Lemma 4.5.2 to $\mathcal{C}$, we show that $\mathcal{C}$ satisfies the hypothesis of Lemma 4.5.2. It is clear that $\mathcal{C}$ is a closed cone. Let $V \in \mathcal{C}$ such that $\min_{x \in S(\Lambda)} \varphi_x(V) = 0$. If $\mathrm{TR}(V) \leq 0$, then we have $V = 0$ by the condition 2. Hence $V \in \mathcal{C} \backslash \{0\}$ with $\min_{x \in S(\Lambda)} \varphi_x(V) = 0$ satisfies $\mathrm{TR}(V) > 0$. Let $V \in \mathcal{C}$ such that $\min_{x \in S(\Lambda)} \varphi_x(V) = k > 0$. Since $V' = V - k\mathrm{Id}$ satisfies $\min_{x \in S(\Lambda)} \varphi_x(V') = 0$, we have $V' \in \mathcal{C}$ and the argument above says that $V' = 0$ or $\mathrm{TR}(V') > 0$. Hence $\mathrm{TR}(V) = \mathrm{TR}(V') + kmn > 0$. Therefore $\mathcal{C}$ satisfies the hypothesis of Lemma 4.5.2, i.e. there exists $\alpha > 0$ such that $\det(\mathrm{Id} + V) > 1$ for any $V \in \mathcal{C}$ with $0 < ||V|| < \alpha$. We fix such $0 < \alpha < 1$.

Let $0 < \beta < \alpha$. Let $V = U^*U - \mathrm{Id}$ for $U \in GL_m(K_{\mathbb{R}})$ and $\mu_V = \min_{x \in S(\Lambda)} \varphi_x(V)$. Since

$$\begin{aligned} 2|\varphi_x(V)| &= 2| < Vx, x > | \\ &= | \ ||(V + \mathrm{Id})x||^2 - ||Vx||^2 - ||x||^2| \\ &\le |(||Vx|| + ||x||)^2 - ||Vx||^2 - ||x||^2| \\ &= 2||Vx|| \cdot ||x|| \\ &\le 2||V||, \end{aligned} \tag{4.1}$$

we have $|\mu_V| < ||V||$ and $c_V = \sqrt{1/(1 + \mu_V)}$ is well-defined if $||V|| < \beta$. Let $\mathcal{U}$ be a neighborhood of $\mathrm{Id} \in GL_m(K_{\mathbb{R}})$ such that $\mathcal{U}$ satisfies Lemma 4.3. Let $\mathcal{W}$ be a neighborhood of $\mathrm{Id} \in GL_m(K_{\mathbb{R}})$ such that $\{V = U^*U - \mathrm{Id} \mid U \in \mathcal{W}\} \subset \{V \in H_m(K_{\mathbb{R}}) \mid ||V|| < \beta\}$ and $\{U' = c_V U \mid U \in \mathcal{W}\} \subset \mathcal{U}$. We show $\gamma_K^{\dagger}(U(\Lambda)) < \gamma_K^{\dagger}(\Lambda)$ for any $U \in \mathcal{W}$. Let $U' = c_V U$ and $V' = U'^* U' - \mathrm{Id} = c_V^2 V - c_V^2 \mu_V \mathrm{Id}$ for $U \in \mathcal{W}$. Since $U' \in \mathcal{U}$ and V' satisfies

$$\begin{aligned} \min_{x \in S(\Lambda)} \varphi_x(V') &= \min_{x \in S(\Lambda)} (c_V^2 < Vx, x > - c_V^2 \mu_V < x, x >) \\ &= c_V^2 \min_{x \in S(\Lambda)} (\varphi_x(V) - \mu_V Q(x)) \\ &= 0, \end{aligned} \tag{4.2}$$

we have $Q(\Lambda) = Q(U'\Lambda)$ by Lemma 4.3. Since $V' \in \mathcal{C}$ and

$$\begin{aligned} ||V'|| &= c_V^4 ||V - \mu_V \mathrm{Id}|| \\ &< \frac{1}{(1 + \mu_V)^2} (||V|| + \mu_V^2 ||\mathrm{Id}||) \\ &< \frac{\beta + \beta^2}{(1 - \beta)^2} \\ &< \alpha, \end{aligned} \tag{4.3}$$

we have $\det(\mathrm{Id} + V') = \det(U')^2 > 1$ by Lemma 4.5.2. Hence we have

$$\begin{aligned} \gamma_K^{\dagger}(U(\Lambda)) &= \gamma_K^{\dagger}(U'(\Lambda)) \\ &= \frac{Q(U'(\Lambda))}{(\det U'(\Lambda))^{1/mn}} \\ &< \frac{Q(\Lambda)}{(\det \Lambda)^{1/mn}} \\ &= \gamma_K^{\dagger}(\Lambda) \end{aligned} \tag{4.4}$$

for any $U \in \mathcal{W}$. $\square$

Proof of Theorem 4.1. We use Proposition 4.1.

Assume that Λ is perfect and eutactic. Let $V \in H_m(K_\mathbb{R})$ such that V satisfies $\min_{x\in S(\Lambda)} \varphi_x(V) = 0$ and $\mathrm{TR}(V) \leq 0$. Since Λ is eutactic, TR is given by a linear combination of φ_x such that $\mathrm{TR} = \sum_{x\in S(\Lambda)} \rho_x\varphi_x$ for some $\rho_x \geq 0$. Hence we have $\varphi_x(V) = 0$ for all $x \in S(\Lambda)$. On the other hand, $\varphi_x(x \in S(\Lambda))$ generates $H_m(K_\mathbb{R})^*$ from the perfection of Λ. Therefore we have $V = 0$.

Conversely, assume that Λ is extreme. First we prove the perfection of Λ. Let $V \in H_m(K_\mathbb{R})$ such that $\varphi_x(V) = 0$ for all $x \in S(\Lambda)$. We may assume that $\mathrm{TR}(V) \leq 0$ by replace V with $-V$ if necessary. Then we have $V = 0$ by Proposition 4.1.2. Hence $\{\varphi_x\}_{x\in S(\Lambda)}$ spans $H_m(K_\mathbb{R})^*$.

Now we prove the eutaxy of Λ. Let $V \in H_m(K_\mathbb{R})$ such that V satisfies $\mathrm{TR}(V) \leq 0$ and $\varphi_x(V) \geq 0$ for all $x \in S(\Lambda)$. Let $k = \min_{x\in S(\Lambda)} \varphi_x(V) \cdot Q(\Lambda)^{-1} \geq 0$ and $V' = V - k\mathrm{Id}$. Then we have $\min_{x\in S(\Lambda)} \varphi_x(V') = \min_{x\in S(\Lambda)}(\varphi_x(V) - kQ(x)) = 0$ and $\mathrm{TR}(V') = \mathrm{TR}(V) - kmn < 0$. Hence we have $V' = 0$ by Proposition 4.1.2. Though we have $V = k\mathrm{Id}$, V satisfies $\mathrm{TR}(V) \leq 0$. Therefore $V = 0$. Applying Lemma 4.1 to linear forms $\varphi_x(x \in S(\Lambda))$ and $-\mathrm{TR}$ defined over $H_m(K_\mathbb{R})$, we can prove the eutaxy of Λ. $\square$

Example. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{-1})$ be a bi-quadratic extension over $\mathbb{Q}$. The Galois group $\mathrm{Gal}(K/\mathbb{Q}) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is given by

$$\sigma_1 : K \ni x + y\sqrt{2} + z\sqrt{-1} + w\sqrt{-2} \longrightarrow x + y\sqrt{2} + z\sqrt{-1} + w\sqrt{-2} \in K$$

$$\sigma_2 : K \ni x + y\sqrt{2} + z\sqrt{-1} + w\sqrt{-2} \longrightarrow x - y\sqrt{2} + z\sqrt{-1} - w\sqrt{-2} \in K$$

$$\sigma_3 : K \ni x + y\sqrt{2} + z\sqrt{-1} + w\sqrt{-2} \longrightarrow x + y\sqrt{2} - z\sqrt{-1} - w\sqrt{-2} \in K$$

$$\sigma_4 : K \ni x + y\sqrt{2} + z\sqrt{-1} + w\sqrt{-2} \longrightarrow x - y\sqrt{2} - z\sqrt{-1} + w\sqrt{-2} \in K$$

where $x, y, z, w \in \mathbb{Q}$. Then σ_3 (resp. σ_4) is equal to a composition of the complex conjugate and σ_1 (resp. σ_2). Hence K is embedded in $K_\mathbb{R} = \mathbb{C}^2$ by $\sigma : K \ni x \hookrightarrow (\sigma_1(x), \sigma_2(x)) \in K_\mathbb{R}$. The ring of integers $\mathcal{O}_K$ of K is given by $\mathcal{O}_K = \mathbb{Z}[(\sqrt{2} + \sqrt{-2})/2]$.

Let $g = (\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}) \in H_1^{++}(K_\mathbb{R}) = \mathbb{R}^2_{>0}$. We consider the $\mathcal{O}_K$-module $\Lambda = g\mathcal{O}_K$.

For $X = ((\sqrt{2+\sqrt{2}})\sigma_1(x), (\sqrt{2-\sqrt{2}})\sigma_2(x)) \in \Lambda$ with $x \in \mathcal{O}_K$, $Q(X)$ is given such that

$$\begin{aligned} Q(X) &= \mathrm{Tr}_{K_\mathbb{R}}(\overline{\sigma_1(x)}(2+\sqrt{2})\sigma_1(x), \overline{\sigma_2(x)}(2-\sqrt{2})\sigma_2(x)) \\ &= 2(\overline{\sigma_1(x)}(2+\sqrt{2})\sigma_1(x) + \overline{\sigma_2(x)}(2-\sqrt{2})\sigma_2(x)). \end{aligned}$$

If $x = a + b\sqrt{2}/2 + c\sqrt{-1} + d\sqrt{-2}/2 \in \mathcal{O}_K$ of $a, b, c, d \in \mathbb{Z}$ and $b + d \in 2\mathbb{Z}$, then we have

$$
\begin{aligned}
Q(X) = 2(a\ b\ c\ d) &\left\{ (2+\sqrt{2}) \begin{pmatrix} 1 & \sqrt{2}/2 & \sqrt{-1} & \sqrt{-2}/2 \\ \sqrt{2}/2 & 1/2 & \sqrt{-2}/2 & \sqrt{-1}/2 \\ -\sqrt{-1} & -\sqrt{-2}/2 & 1 & \sqrt{2}/2 \\ -\sqrt{-2}/2 & -\sqrt{-1}/2 & \sqrt{2}/2 & 1/2 \end{pmatrix} \right. \\
&\left. +(2-\sqrt{2}) \begin{pmatrix} 1 & -\sqrt{2}/2 & \sqrt{-1} & -\sqrt{-2}/2 \\ -\sqrt{2}/2 & 1/2 & -\sqrt{-2}/2 & \sqrt{-1}/2 \\ -\sqrt{-1} & \sqrt{-2}/2 & 1 & -\sqrt{2}/2 \\ \sqrt{-2}/2 & -\sqrt{-1}/2 & -\sqrt{2}/2 & 1/2 \end{pmatrix} \right\} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \\
&= 2(a\ b\ c\ d) \begin{pmatrix} 4 & 2 & 4\sqrt{-1} & 2\sqrt{-1} \\ 2 & 2 & 2\sqrt{-1} & 2\sqrt{-1} \\ -4\sqrt{-1} & -2\sqrt{-1} & 4 & 2 \\ -2\sqrt{-1} & -2\sqrt{-1} & 2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \\
&= 2(2a+b)^2 + 2b^2 + 2(2c+d)^2 + 2d^2.
\end{aligned}
$$

Hence the set of minimal vectors of Λ is given by

$$
\begin{aligned}
S(\Lambda) = &\left\{ g\sigma(x) \;\middle|\; x = \pm 1, \pm\sqrt{-1}, \pm(1-\sqrt{2}), \pm(\sqrt{-1}-\sqrt{-2}), \right. \\
&\pm\frac{\sqrt{2}}{2}(1 \pm \sqrt{-1}), \pm\frac{\sqrt{2}}{2}(1-\sqrt{2} \pm \sqrt{-1}), \pm\frac{\sqrt{2}}{2}(1 \pm (\sqrt{-2}-\sqrt{-1})), \\
&\left. \pm\frac{\sqrt{2}}{2}(1-\sqrt{2} \pm (\sqrt{-2}-\sqrt{-1})) \right\} \\
= &\left\{ g\sigma(x) \;\middle|\; \begin{array}{c} x = e^{k\pi\sqrt{-1}/4},\ (\sqrt{2}-1)e^{k\pi\sqrt{-1}/4}, \\ \left(\sqrt{2-\sqrt{2}}\right) e^{(2k+1)\pi\sqrt{-1}/8}, \end{array} \quad \text{for } 0 \le k \le 7 \right\},
\end{aligned}
$$

where orders of signs are arbitrary. Especially we have $Q(\Lambda) = 8$.

For $X = g\sigma(x) \in S(\Lambda)$ and $(v_1, v_2) \in H_1(K_{\mathbb{R}}) = \mathbb{R}^2$, φ_X is given by

$$\varphi_X(v_1, v_2) = 2(2+\sqrt{2})v_1\overline{\sigma_1(x)}\sigma_1(x) + 2(2-\sqrt{2})v_2\overline{\sigma_2(x)}\sigma_2(x).$$

Hence we can describe φ_X as follows:

(a) $x = e^{k\pi\sqrt{-1}/4}$ $\Rightarrow \varphi_X(v_1, v_2) = (4+2\sqrt{2})v_1 + (4-2\sqrt{2})v_2$

(b) $x = (\sqrt{2}-1)e^{k\pi\sqrt{-1}/4}$ $\Rightarrow \varphi_X(v_1, v_2) = (4-2\sqrt{2})v_1 + (4+2\sqrt{2})v_2$

(c) $x = (\sqrt{2-\sqrt{2}})e^{(2k+1)\pi\sqrt{-1}/8}$ $\Rightarrow \varphi_X(v_1, v_2) = 4v_1 + 4v_2.$

Since (a) and (b) span the dual space of $H_1(K_{\mathbb{R}})$ as a $\mathbb{R}$-vector space, Λ is perfect. Moreover Λ is obviously eutactic.

5. Finiteness of perfect $\mathcal{O}_K$-lattices

In this section, we prove that the number of similar isometry classes of perfect $\mathcal{O}_K$-lattices in $K_{\mathbb{R}}^m$ is finite. We keep the notations in Section 3.

Let $\Lambda_1, \cdots, \Lambda_h$ be the lattices defined in Section 2. We fix a $\Lambda_0 \in \{\Lambda_1, \cdots, \Lambda_h\}$. The discrete group $GL(\Lambda_0)$ acts on $H_m^{++}(K_{\mathbb{R}})$ as usual, i.e. $H \cdot \gamma = \gamma^* H \gamma$ for $\gamma \in GL(\Lambda_0)$ and $H \in H_m^{++}(K_{\mathbb{R}})$. Let $\alpha > 1$ be a sufficient large constant. We define the Siegel set $\mathcal{S}_\alpha$ of $GL_m(K_{\mathbb{R}})$ by

$$\mathcal{S}_\alpha = \left\{ T^*DT \in GL_m(K_{\mathbb{R}}) \;\middle|\; \begin{array}{c} D_k = \begin{pmatrix} d_{k,1} & & \\ & \ddots & \\ & & d_{k,m} \end{pmatrix}, \\ T_k = \begin{pmatrix} 1 & & t_{ij}^{(k)} \\ & \ddots & \\ & & 1 \end{pmatrix}, \\ d_{k,i} > 0,\ \dfrac{d_{k,i}}{d_{k',i}} < \alpha,\ \dfrac{d_{k,i}}{d_{k,i+1}} < \alpha,\ |t_{ij}^{(k)}| < \alpha \\ (1 \le i \le j \le m, 1 \le k, k' \le r+s) \end{array} \right\}.$$

For a given $B \in GL_m(K_{\mathbb{R}})$, we set

$$\Omega_{\alpha,B} = \{B^*T^*DTB \mid T^*DT \in \mathcal{S}_\alpha\}.$$

Then Humbert's reduction theory says that there exists a finite subset $\{B_1, \cdots, B_\kappa\}$ of $GL_m(K)$ such that $H_m^{++}(K_{\mathbb{R}}) = \cup_{i=1}^{\kappa} \Omega_{\alpha,B_i} \cdot GL(\Lambda_0)$ (See [2] and [8], Theorem 18.5).

Theorem 5.1. *The number of similar isometry classes of perfect $\mathcal{O}_K$-lattices in $K_{\mathbb{R}}^m$ is finite.*

Proof. Let Λ be a perfect $\mathcal{O}_K$-lattice with $Q(\Lambda) = 1$. Then there exists $g \in GL_m(K_{\mathbb{R}})$ and $\Lambda_0 \in \{\Lambda_1, \cdots, \Lambda_h\}$ such that $\Lambda = g\Lambda_0$. Since $g^*g \in H_m^{++}(K_{\mathbb{R}})$, there exists some $\alpha > 1$ and $B \in \{B_1, \cdots, B_\kappa\}$ such that $g^*g \in \Omega_{\alpha,B} \cdot GL(\Lambda_0)$. Let $g^*g = \gamma^*B^*T^*DTB\gamma$ for $\gamma \in GL(\Lambda_0)$ and $T^*DT \in \mathcal{S}_\alpha$. We put $D_{r+s+\ell} = D_{r+\ell}$ and $T_{r+s+\ell} = (\overline{t_{ij}}^{(r+\ell)})$ for $1 \le \ell \le s$. Let $x \in S(\Lambda)$ be a minimum vector. Then x is denoted by $x = gx_0$ for some $x_0 \in \Lambda_0$ and we have

$$\begin{aligned} Q(x) =& Q(\sqrt{D}TB\gamma x_0) = \sum_{k=1}^{n} ||\sqrt{D_k}T_k\sigma_k(B)\sigma_k(\gamma x_0)||^2 \\ \ge& \sum_{k=1}^{n} ||(\sqrt{D_k}T_k\sigma_k(B))^{-1}||^{-2}||\sigma_k(y_0)||^2 \qquad (5.1) \\ \ge& \sum_{k=1}^{n} ||\sqrt{D_k}^{-1}||^{-2}||T_k^{-1}||^{-2}||\sigma_k(B)^{-1}||^{-2}||\sigma_k(y_0)||^2, \end{aligned}$$

where $y_0 = \gamma x_0 \in \Lambda_0$. Since all entries of $T_k^{-1} = (t_{ij}^{\prime(k)})$ are bounded by a constant $\alpha(\alpha+1)^{m-2}$, $||T_k^{-1}||^2$ is bounded as

$$
\begin{aligned}
||T_k^{-1}||^2 &= \sup_{y\neq 0} \frac{||T_k^{-1} y||^2}{||y||^2} = \sup_{y\neq 0} ||T_k^{-1} \frac{y}{||y||}||^2 = \sup_{||z||=1} ||T_k^{-1} z||^2 \\
&= \sup_{||z||=1} \sum_{i=1}^{m} |\sum_{j=i}^{m} t_{ij}^{\prime(k)} z_j|^2 \\
&\leq \sum_{i=1}^{m} (m-i+1)^2 \alpha^2 (\alpha+1)^{2m-4} \\
&= \frac{1}{6} m(m+1)(2m+1)\alpha^2(\alpha+1)^{2m-4},
\end{aligned} \tag{5.2}
$$

where $z = (z_j)$. We put $C_T = 6^{-1} m(m+1)(2m+1)\alpha^2(\alpha+1)^{2m-4}$. And more, since $\dfrac{d_{k,1}}{d_{k,j}} = \dfrac{d_{k,1}}{d_{k,2}} \cdots \dfrac{d_{k,j-1}}{d_{k,j}} \leq \alpha^{j-1} \leq \alpha^m$, $||\sqrt{D_k}^{-1}||^2$ is also bounded as

$$
\begin{aligned}
||\sqrt{D_k}^{-1}||^2 &= \sup_{y\neq 0} \frac{||y||^2}{||\sqrt{D_k} y||^2} = \sup_{y\neq 0} \frac{||y||^2}{\sum_{j=1}^{m} d_{k,j} y_j \overline{y_j}} \\
&\leq \sup_{y\neq 0} \frac{||y||^2}{\min_j d_{k,j} ||y||^2} = (\min_j d_{k,j})^{-1} \\
&\leq d_{k,1}^{-1} \alpha^m.
\end{aligned} \tag{5.3}
$$

We put $\beta = \max(||\sigma_k(B_1)^{-1}||, \cdots, ||\sigma_k(B_\kappa)^{-1}||)$.

Let $e_1 = {}^t(1, 0, \cdots, 0) \in K^m$. Then there exists $\lambda_B \in \mathcal{O}_K$ for each $B \in \{B_1, \cdots, B_\kappa\}$ such that $e_0 = \gamma^{-1}(\lambda_B B^{-1} e_1)$ is contained in Λ_0 and e_0 satisfies $Q(ge_0) \leq Q(\lambda_B e_1) \sum_{k=1}^{n} d_{k,1}$. We put $c = \max_{1\leq i\leq\kappa} Q(\lambda_{B_i} e_1)$. Then we have

$$
\begin{aligned}
Q(x) &\geq \sum_{k=1}^{n} ||\sqrt{D_k}^{-1}||^{-2} ||T_k^{-1}||^{-2} ||\sigma_k(B)^{-1}||^{-2} ||\sigma_k(y_0)||^2 \\
&\geq \sum_{k=1}^{n} d_{k,1} \alpha^{-m} \cdot C_T^{-1} \beta^{-2} ||\sigma_k(y_0)||^2 \\
&\geq \min_k d_{k,1} \left(\sum_{k=1}^{n} \alpha^{-m} C_T^{-1} \beta^{-2} ||\sigma_k(y_0)||^2 \right) \\
&\geq n^{-1} \left(\sum_{k=1}^{n} \alpha^{-1} d_{k,1} \right) \left(\sum_{k=1}^{n} \alpha^{-m} C_T^{-1} \beta^{-2} ||\sigma_k(y_0)||^2 \right) \\
&\geq C_{K,m}^{-1} Q(ge_0) \left(\sum_{k=1}^{n} ||\sigma_k(y_0)||^2 \right),
\end{aligned} \tag{5.4}
$$

where $C_{K,m} = n\alpha^{m+1} C_T \beta^2 c$ is a constant which depends only on K and m. Let $S_0 = \{z \in \Lambda_0 \mid \sum_{k=1}^n ||\sigma_k(z)||^2 \leq C_{K,m}\}$, which is a finite subset of Λ_0. Then $y_0 \in S_0$ since $\sum_{k=1}^n ||\sigma_k(y_0)||^2$ is bounded as

$$\sum_{k=1}^n ||\sigma_k(y_0)||^2 \leq C_{K,m} \frac{Q(x)}{Q(ge_0)} \leq C_{K,m}.$$

Therefore $\gamma g^{-1} S(\Lambda)$, and hence $g^{-1} S(\Lambda)$, is contained S_0. By the proof of Theorem 3.1, any perfect $\mathcal{O}_K$-lattice is uniquely determined by its shortest vectors and its minimum. Since the possibility of subsets of S_0 is finite, the number of perfect $\mathcal{O}_K$-lattices up to similarity and isometry is finite. □

Remark. Theorem 5.1 was proved by Koecher ([3], Section 9.10) in the case of $\Lambda_0 = \mathcal{O}_K^m$. It is possible to apply Koecher's method to a general Λ_0. Our proof is different from Koecher's. We sketch Koecher's method in the following.

We write ξ for the constant $2^{mn} \rho_{mn}^{-1} \Delta_K^{m/2}$, where ρ_{mn} is the volume of the unit ball of dimension mn and Δ_K is the absolute value of the discriminant of K. We take $a_1, \cdots, a_s \in \mathcal{O}_K$ such that $1 \leq N_{K/\mathbb{Q}}(a_1) < \cdots < N_{K/\mathbb{Q}}(a_s) \leq \xi$ and s is as big as possible. We define an equivalent relation in the set $\{X \in M_m(\mathcal{O}_K) \mid \det X = a_i \mathcal{O}_K^\times\}$ for each i as follows : $X_1 \sim X_2$ if $X_1 - X_2 \in a_i M_m(\mathcal{O}_K)$. Let $B(a_i)$ be a complete system of representative of the quotient set $\{X \in M_m(\mathcal{O}_K) \mid \det X = a_i \mathcal{O}_K^\times\}/\sim$, and let $B = \bigcup_{i=1}^s B(a_i)$. Define $A = \{x \in \mathcal{O}_K^m \mid x$ is a column vector of some matrix in $B\}$ and $\tilde{A} = \{xx^* \mid x \in A\}$. We consider a subset $N \subset \tilde{A}$ such that $a_N = \sum_{X \in N} X$ is positive definite. Since each $B(a_i)$ is a finite set, the number of such subset N is finite.

Let $V = \{v \in H_m^{++}(K_\mathbb{R}) \mid v$ is perfect with minimum $1\}$. For $v \in V$, there exist linear independent minimal vectors $x_1, \cdots, x_m \in \mathcal{O}_K^m$ such that $1 \leq |N_{K_\mathbb{R}/\mathbb{R}}(\det(x_1, \cdots, x_m))| \leq \xi$. By Koecher's Lemma ([3], Lemma 13), there exists a unimodular matrix u such that $u(x_1, \cdots, x_m) \in B$. This induces $(ux_i)(ux_i)^* \in \tilde{A}$ for all $1 \leq i \leq m$. Since $a_v = \sum_{i=1}^m (ux_i)(ux_i)^*$ is a positive definite, $a_v = a_N$ for some $N \subset \tilde{A}$. Finiteness of N implies that there exists a self-adjoint matrix $d \in H_m^{++}(K_\mathbb{R})$ such that $a_v - d \in H_m^{++}(K_\mathbb{R})$ for all $v \in V$. This d is independent of the choice of $v \in V$. Then we have

$$\begin{aligned} m &= \sum_{i=1}^m < vx_i, x_i > \\ &= \mathrm{Tr}_{K_\mathbb{R}/\mathbb{R}} \mathrm{Tr}((u^{-1*} v u^{-1}) a_V) \qquad (5.5) \\ &> \mathrm{Tr}_{K_\mathbb{R}/\mathbb{R}} \mathrm{Tr}((u^{-1*} v u^{-1}) d) \\ &> \rho(d) \mathrm{TR}((u^{-1*} v u^{-1})^2), \end{aligned}$$

where $\rho(d)$ is a constant which depends only on K and m. Since $\mathrm{TR}((u^{-1*}vu^{-1})^2)$ is bounded by a constant and $V \subset H_m^{++}(K_{\mathbb{R}})$ is discrete, the number of V is finite up to the action by unimodular matrix u.

Acknowledgments

We thank Professor Takao Watanabe for suggesting this problem and for many discussions. We would like to thank the referees for pointing out Koecher's papers.

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