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Homology for irregular connections

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Résumé. Nous définissons sur une courbe algébrique l'homologie à valeurs dans une connexion avec des points singuliers éventuellement irréguliers, généralisant ainsi l'homologie à valeurs dans le système local sous-jacent pour une connexion avec points singuliers réguliers. L'intégration définit alors un accouplement parfait entre la cohomologie de de Rham à valeurs dans la connexion et l'homologie à valeurs dans la connexion duale.

Abstract. Homology with values in a connection with possibly irregular singular points on an algebraic curve is defined, generalizing homology with values in the underlying local system for a connection with regular singular points. Integration defines a perfect pairing between de Rham cohomology with values in the connection and homology with values in the dual connection.

0. Introduction

Consider the following formulas, culled, one may imagine, from a textbook on calculus:

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$(e^{2\pi i s} - 1)\Gamma(s) = (e^{2\pi i s} - 1)\int_0^{\infty} e^{-t} t^s \frac{dt}{t} \qquad \text{Gamma function}$$

$$J_n(z) = \frac{1}{2\pi i}\int_{\{|u|=\epsilon\}} \exp(\frac{z}{2}(u - \frac{1}{u}))\frac{du}{u^{n+1}} \qquad \text{Bessel function.}$$

These are a few familiar examples of periods associated to connections with irregular singular points on Riemann surfaces. Curiously, though of course such integrals have been studied for 200 years or so, and mathematicians in recent years have developed a powerful duality theory for holonomic $\mathcal{D}$-modules (for dimension 1, which is the only case we will consider, cf. [4],chap. IV, and [5]), it is not easy from the literature to interpret such integrals as periods arising from a duality between homological cycles and differential forms. A homological duality of this sort is well understood for differential equations with regular singular points, and for special rank 1 differential equations [2]. Our purpose in this note is to develop a similar theory in the irregular case. Of course, most of the "heavy lifting" was done by Malgrange op. cit. We hope, in reinterpreting his theory, to better understand relations between irregular connections and wildly ramified ℓ-adic sheaves. There are striking relations between ϵ-factors for ℓ-adic sheaves on curves over finite fields and determinants of irregular periods [8] which merit further study. Finally, relations between irregular connections and the arithmetic theory of motives remain mysterious.

Let X be a smooth, compact, connected algebraic curve (Riemann surface) over $\mathbb{C}$. Let $D = \{x_1, \ldots, x_n\} \subset X$ be a non-empty, finite set of points (which we also think of as a reduced effective divisor), and write $U := X \setminus D \overset{j}{\hookrightarrow} X$. Let E be a vector bundle on X, and suppose given a connection with meromorphic poles on D

$$\nabla : E \to E \otimes \omega(*D).$$

Here ω is the sheaf of holomorphic 1-forms on X and $*D$ refers to meromorphic poles on D. Unless otherwise indicated, we work throughout in the analytic topology. The de Rham cohomology $H^*_{DR}(X \setminus D; E, \nabla)$ is the cohomology of the complex of sections

$$\Gamma(X, E(*D)) \xrightarrow{\nabla} \Gamma(X, E \otimes \omega(*D)) \tag{0.1}$$

placed in degrees 0 and 1. These cohomology groups are finite dimensional [1], Proposition 6.20, (i).

Let $E^\vee$ be the dual bundle, and let $\nabla^\vee$ be the dual connection, so

$$d\langle e, f^\vee \rangle = \langle \nabla(e), f^\vee \rangle + \langle e, \nabla^\vee(f^\vee) \rangle. \tag{0.2}$$

Define $\mathcal{E} = \ker(\nabla)$, and $\mathcal{E}^\vee = \ker(\nabla^\vee)$ to be the corresponding local systems of flat sections on U. We want to define homology with values in these local systems, or more precisely with values in associated cosheaves on X. For $x \in X \setminus D$, $\mathcal{E}_x$ will denote the stalk of $\mathcal{E}$ at x. Define the co-stalk at $0 \in D$

$$\mathcal{E}_0 := \mathcal{E}_x / (1 - \sigma)\mathcal{E}_x \tag{0.3}$$

where $x \neq 0$ is a nearby point, and σ is the local monodromy about 0. We write $\mathcal{C}_n = \mathcal{C}_n(E, \nabla)$ for the group of n-chains with values in $\mathcal{E}$ and rapid decay near 0. Write Δ^n for the n-simplex and $b \in \Delta^n$ for its barycenter. Thus, $\mathcal{C}_n(E, \nabla)$ is spanned by elements $c \otimes \epsilon$ with $c : \Delta^n \to X$ and $\epsilon \in \mathcal{E}_{c(b)}$, where $b \in \Delta^n$ is the barycenter. We assume $c^{-1}(0) =$ union of faces $\subset \Delta^n$ and that ϵ has *rapid decay* near D. This is no condition if $D \cap c(\Delta^n) = \emptyset$. If $0 \in D \cap c(\Delta^n)$, we take e_i a basis for E near 0 and write $\epsilon = \sum f_i c^*(e_i)$. Let z be a local parameter at 0 on Δ. We require that for all $N \in \mathbb{N}$, constants $C_N > 0$ exist with $|f_i(z)| \leq C_N |z|^N$ on $\Delta^n \setminus c^{-1}(0)$. Note that if ∇ has logarithmic poles in one point, then rapid decay implies vanishing. Thus in this case, we deal with the sheaf $j_! \mathcal{E}$, where $j : X \setminus D \to X$.

There is a natural boundary map

$$\partial : \mathcal{C}_n(E, \nabla) \to \mathcal{C}_{n-1}(E, \nabla); \quad \partial(c \otimes \epsilon) = \sum (-1)^j c_j \otimes \epsilon_j \tag{0.4}$$

where c_j are the faces of c. Note if b_j is the barycenter of the j-th face and $c(b_j) \neq 0$, c determines a path from $c(b)$ to $c(b_j)$ which is canonical upto homotopy on $\Delta \setminus \{0\}$. (As a representative, one can take $c[b_j, b]$, the image of the straight line from b to b_j. By assumption, $c^{-1}(0)$ is a union of faces, so it does not meet the line.) Thus $\epsilon \in \mathcal{E}_{c(b)}$ determines $\epsilon_j \in \mathcal{E}_{c(b_j)}$. Similarly for $0 \in D$, if $c(b_j) = 0$ there is corresponding to ϵ a unique $\epsilon_j \in \mathcal{E}_0$ because we have taken coinvariants. If $c : \Delta^n \to D$ is a constant simplex, there is no rapid decay condition.

It is straightforward to compute that $\partial \circ \partial = 0$. Consider $c \otimes \epsilon$. If $c(b) = 0$, where $b \in \Delta^2$ is the barycentre, then $c(\Delta^2) = 0$ and $\epsilon = \epsilon_i = (\epsilon_i)_j \in \mathcal{E}_0$ for all i and j involved, thus the condition is trivially fulfilled. If not, and some $c(b_i) = 0$, then $(\epsilon_j)_i = (\epsilon_i)_j \in \mathcal{E}_0$ for all j, and if all $c(b_i) \neq 0$, then one has by unique analytic continuation in $c(\Delta^2)$ the relation $(\epsilon_i)_j = (\epsilon_j)_i \in \mathcal{E}_{\mathrm{edge}_{ij}}$ for all i, j, if $\mathrm{edge}_{ij} \neq 0$, else in $\mathcal{E}_0$.

We define

$$H_*(X, D; E^\vee, \nabla^\vee) := H_*\Big(\mathcal{C}_*(X; E^\vee, \nabla^\vee)/\mathcal{C}_*(D; E^\vee, \nabla^\vee)\Big). \tag{0.5}$$

(The growth condition means this depends on more than just the topological sheaf $\mathcal{E}^\vee$, so we keep $E^\vee, \nabla^\vee$ in the notation.)

We now define a pairing

$$(\ ,\ ) : H^*_{DR}(X \setminus D; E, \nabla) \times H_*(X, D; E^\vee, \nabla^\vee) \to \mathbb{C}; \quad * = 0, 1 \tag{0.6}$$

by integrating over chains in the following manner. For $* = 0$, then $H_0(X, D; E^\vee, \nabla^\vee)$ is generated by sections of the dual local system $\mathcal{E}^\vee$ in points $\in X$ while $H^0_{DR}(X \setminus D; E, \nabla)$ is generated by global flat sections in $\mathcal{E}$ with moderate growth. So one can pair them. For $* = 1$, since $D \neq \emptyset$, then

$$H^1_{DR}(X \setminus D; E, \nabla) = H^0(X, \omega \otimes E(*D))/\nabla H^0(X, E(*D)),$$

and since classes $c \otimes \epsilon$ generating $H_0(X, D; E^\vee, \nabla^\vee)$ have rapid decay, the integral $\int_c < f_i c^*(e_i), \alpha >$ is convergent, where $\alpha \in H^0(X, \omega \otimes E(*D))$ and $< >$ is the duality between $E^\vee$ and E.

The rest of the note is devoted to the proof of the following theorem.

Theorem 0.1. *The process of integrating forms over chains is compatible with homological and cohomological equivalences and defines a perfect pairing of finite dimensional vector complex spaces*

$$(\ ,\ ) : H^*_{DR}(X \setminus D; E, \nabla) \times H_*(X, D; E^\vee, \nabla^\vee) \to \mathbb{C}; \quad * = 0, 1.$$

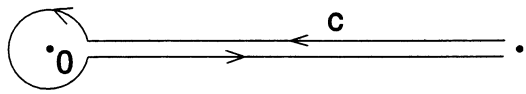

FIGURE 1. $c \otimes e^{-t}t^s$ represents a class in H_1

Example 0.2. (i). If ∇ has regular singular points, there are no rapidly decaying flat sections, so $H_*(X, D; E^\vee, \nabla^\vee) \cong H_*(X \setminus D; \mathcal{E}^\vee)$. Also, $H^*_{DR}(X \setminus D; E, \nabla) \cong H^*(U, \mathcal{E})$ (cf. [1], Théorème 6.2), and the theorem becomes the classical duality between homology and cohomology.
(ii). Suppose $X = \mathbb{P}^1$, $D = \{0, \infty\}$. Let $E = \mathcal{O}_{\mathbb{P}^1}$ with connection $\nabla(1) = -dt + s\frac{dt}{t}$, for some $s \in \mathbb{C} \setminus \{0, 1, 2, \ldots\}$. Then $\mathcal{E} \subset E_U = \mathcal{O}_U$ is the trivial local system spanned by $e^t t^{-s}$, so $\mathcal{E}^\vee \subset E_U^\vee = \mathcal{O}_U$ is spanned by $e^{-t}t^s$. We consider the pairing $H^1_{DR} \times H_1 \to \mathbb{C}$ from theorem 0.1. Note first that H^1_{DR} has dimension 1, spanned by $\frac{dt}{t}$. This can either be checked directly from (0.1), using

$$\nabla(t^p) = ((p+s)t^{p-1} - t^p)dt,$$

or by showing the de Rham cohomology is isomorphic to the hypercohomology of the complex $\mathcal{O}_{\mathbb{P}^1} \xrightarrow{\nabla} \omega((0) + 2(\infty))$, which is easily computed. To compute $H_1(X, D; E^\vee, \nabla^\vee)$, the singularity at 0 is regular, so there are no non-constant, rapidly decaying chains at 0. The section $\epsilon^\vee := e^{-t}t^s$ of $\mathcal{E}^\vee$ is rapidly decaying on the positive real axis near ∞, so the chain $c \otimes \epsilon^\vee$ in fig. 1 above represents a 1-cycle. We have

$$(c \otimes e^{-t}t^s, \frac{dt}{t}) = (e^{2\pi i s} - 1) \int_0^\infty e^{-t}t^s \frac{dt}{t}$$

which is a variant of Hankel's formula (see [10], p. 245).
(iii). Let X, D, E be as in (ii), but take $\nabla(1) = \frac{1}{2}(d(zu) - d(\frac{z}{u}))$ for some $z \in \mathbb{C} \setminus \{0\}$. Here the connection has pole order 2 at 0 and ∞ and it has trivial monodromy. Arguing as above, one computes $\dim H^1_{DR} = 2$, generated by $u^p du$, $p \in \mathbb{Z}$, with relations $u^p du = -\frac{2p}{z}u^{p-1}du - u^{p-2}du$. The Gauß-Manin connection on this group is

$$\nabla_{GM}(u^p du) = \frac{1}{2}(u^{p+1} - u^{p-1})du \wedge dz.$$

Assume $\mathrm{Im}(z) > 0$. Then the vector space $H_1(\mathbb{P}^1, \{0, \infty\}; E^\vee, \nabla^\vee)$ is generated by

$$\{|u| = 1\} \otimes \exp(\frac{1}{2}z(u - \frac{1}{u})), \quad \text{and} \quad [0, i\infty] \otimes \exp(\frac{1}{2}z(u - \frac{1}{u})).$$

(If $\mathrm{Im}(z) \not> 0$, then the second path must be modified.) The integrals

$$J_n(z) := \int_{\{|u|=1\}} \exp(\frac{1}{2}z(u - \frac{1}{u}))\frac{du}{u^{n+1}};$$
$$H_n(z) := \int_0^{i\infty} \exp(\frac{1}{2}z(u - \frac{1}{u}))\frac{du}{u^{n+1}}$$

are periods and satisfy the Bessel differential equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0$$

The function J_n is entire. To show that H_n is linearly independent of J_n, it will then be sufficient to show that H_n is unbounded on the positive part of the imaginary axis $\mathrm{Re}(z) = 0$ as $z \to 0$. Making the coordinate change $v = \frac{1}{u}$, and replacing y by $\frac{1}{2}y$ one is led to show that $E_n(y) = \int_0^\infty \exp(-y(v + \frac{1}{v}))\frac{dv}{v^{n+1}}$ is unbounded for $y > 0, y \to 0$. Writing $E_n(v) = \int_0^1 + \int_1^\infty$, and making the change of variable $v \to \frac{1}{v}$ in the integral $\int_0^1$, one obtains

$$E_n(y) = \int_1^\infty \exp(-y(v + \frac{1}{v}))(\frac{1}{v^{n+1}} + v^{n-1})dv$$
$$\geq \int_1^\infty \exp(-2yv)(\frac{1}{v^{n+1}} + v^{n-1})dv.$$

For $|n| \geq 1$, then this expression is $\geq \int_1^\infty \exp(-2yv)dv$ which is obviously unbounded. For $n = 0$, one has

$$E_0(y) \geq 2\int_1^\infty \exp(-2yv)\frac{dv}{v}$$
$$\geq 2\int_{2y}^\infty \exp(-v)\frac{dv}{v}$$
$$\geq 2\int_{2y}^1 \exp(-v)\frac{dv}{v},$$

where in the last inequality, we have assumed that $2y \leq 1$. This last integral is, up to something bounded, equal to $2\int_{2y}^1 \frac{dv}{v} = -2\log(2y)$, which is unbounded, as $y > 0, y \to 0$.

Usually, for integers $n \in \mathbb{Z}$, one considers J_n as one standard solution, but not H_n (see [10], p.371). Finally, to get Bessel functions for non-integral values of n, one may consider the connection $\nabla(1) = \frac{1}{2}(d(zu) - d(\frac{z}{u})) - n\frac{du}{u}$.

1. Chains

Let $D = \{x_1, \dots, x_n\}$ be as above, and let Δ_i be a small disk about x_i for each i. Let δ_i be the boundary circle. Define

$$(1.1)\quad H_*(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla) \\ = H_*\Big(\mathcal{C}_*(\Delta_i; E, \nabla)/(\mathcal{C}_*(\delta_i; E, \nabla) + \mathcal{C}_*(\{x_i\}; E, \nabla)\Big)$$

(Note, for a set like δ_i which is closed and disjoint from D, our chains coincide with the usual topological chains with values in the local system $\mathcal{E}$. The group $\mathcal{C}_*(\{x_i\}; E, \nabla)$ consists of constant chains $c : \Delta^n \to \{x_i\}$ with values in

$$\mathcal{E}_{x_i} := \mathcal{E}_x/(1 - \mu_i)\mathcal{E}_x$$

for some x near x_i as in (0.3), where μ_i is the local monodromy around x_i.) In the following theorem, $H_*(U, \mathcal{E})$ is the standard homology associated to the local system on $U = X \setminus D$.

Theorem 1.1. *With notation as above, there is a long exact sequence*

$$(1.2)\quad 0 \to H_1(U, \mathcal{E}) \to H_1(X, D; E, \nabla) \to \oplus_i H_1(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla) \\ \to H_0(U, \mathcal{E}) \to H_0(X, D; E, \nabla) \to 0.$$

Proof. Let $\mathcal{C}_* := \mathcal{C}_*(X; E, \nabla)/\mathcal{C}_*(D; E, \nabla)$ be the complex calculating $H_*(X, D; E, \nabla)$, and let

$$\mathcal{C}_*(U) \subset \mathcal{C}_*$$

be the subcomplex calculating $H_*(U, \mathcal{E})$, i.e. the subcomplex of chains whose support is disjoint from D. Of course, one has $\mathcal{C}_*(U; E, \nabla) = \mathcal{C}_*(U; \mathcal{E})$, which justifies the notation.

Write $\mathcal{B} = \mathcal{C}_*/\mathcal{C}_*(U)$. There is an evident map of complexes

$$(1.3)\qquad \psi : \oplus_i \mathcal{C}_*(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla) \to \mathcal{B}$$

which must be shown to be a quasi-isomorphism. Let

$$\mathcal{B}(i) = \psi(\mathcal{C}_*(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla)) = \\ \mathcal{C}_*(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla)/\mathcal{C}_*(\Delta_i \setminus \{x_i\}; \mathcal{E}) \subset \mathcal{B}.$$

Obviously the map $\alpha : \oplus_i \mathcal{B}(i) \hookrightarrow \mathcal{B}$ is an inclusion. We claim first that α is a quasi-isomorphism. To see this, note that all these complexes admit subdivision maps subd which are homotopic to the identity. Given a chain $c \in \mathcal{B}$, there exists an N such that $\mathrm{subd}^N(c) \in \oplus \mathcal{B}(i)$. Taking c with $\partial c = 0$, it follows that $\oplus H_*(\mathcal{B}(i))$ surjects onto $H_*(\mathcal{B})$. If $\alpha(x) = \partial y$, we choose N such that $\mathrm{subd}^N(y) = \alpha(z)$. Since α is injective and commutes with subd, it follows that α is injective on homology as well, so α is a quasi-isomorphism.

It remains to show the surjective map of complexes

$$\beta : \mathcal{C}_*(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla) \to \mathcal{B}(i)$$

is a quasi-isomorphism. The kernel of β is

$$\mathcal{C}_*(\Delta_i \setminus \{x_i\}; \mathcal{E})/\mathcal{C}_*(\delta_i; \mathcal{E}),$$

which is acyclic as $\delta_i \hookrightarrow \Delta_i \setminus \{x_i\}$ admits an evident homotopy retract.

The next point is to show

$$H_*(\Delta_i, \delta_i \cup \{x_i\}; E, \nabla) = (0); \ i = 0, 2. \tag{1.4}$$

The assertion for H_0 is easy because any point y in $\Delta_i \setminus \{x_i\}$ can be attached to δ_i by a radial path r not passing through x_i. Then $\epsilon \in \mathcal{E}_y$ extends uniquely to ϵ on r and $\partial(r \otimes \epsilon) \equiv y \otimes \epsilon$ mod chains on δ_i. Vanishing in (1.4) when $i = 2$ will be proved in a sequence of lemmas. For convenience we drop the subscript i and replace x_i with 0.

Lemma 1.2. *Let $\ell \subset \Delta$ be a radial line meeting δ at p. Let $\mathcal{E}_\ell$ be the space of sections of the local system along $\ell \setminus \{0\}$ with rapid decay at 0. Then*

$$H_*(\ell, \{0, p\}; E, \nabla) \cong \begin{cases} 0 & * \neq 0 \\ \mathcal{E}_\ell & * = 1. \end{cases}$$

Proof of lemma. Let $\mathcal{C}_*(\ell)$ be the complex of chains calculating this homology, and let $\mathcal{C}_*(\ell \setminus \{0\}) \subset \mathcal{C}_*(\ell)$ be the subcomplex of chains not meeting 0. Then $\mathcal{C}_*(\ell \setminus \{0\})$ is contractible, and

$$\mathcal{C}_*(\ell)/\mathcal{C}_*(\ell \setminus \{0\}) \cong (\mathbf{C}_*(\ell)/\mathbf{C}_*(\ell \setminus \{0\})) \otimes \mathcal{E}_\ell$$

where $\mathbf{C}_*$ denotes classical topological chains. The result follows. □

One knows from the theory of irregular connections in dim 1 [4] that $\Delta \setminus \{0\}$ can be covered by open sectors $V \subsetneq \Delta$ such than

$$E, \nabla|_V \cong \oplus_i (L_i \otimes M_i) \tag{1.5}$$

where L_i is rank 1 and M_i has a regular singular point. Let $W \subset V \cup \{0\}$ be a smaller closed sector with outer boundary $\delta_W = \delta \cap W$ and radial sides ℓ_1, ℓ_2. Recall the Stokes lines are radial lines where the horizontal sections of the L_i shift from rapid decay to rapid growth. We assume W contains at most one Stokes line, and that ℓ_1, ℓ_2 are not Stokes lines. Writing $W = W_1 \cup W_2$, where W_i are even smaller sectors, each of which containing the Stokes line if there is one, one may think of the following lemma as a Mayer-Vietoris sequence.

Lemma 1.3. *With notation as above, let w be a basepoint in the interior of W. Then*

$$H_*(W, \delta_W \cup \{0\}; E, \nabla) \cong \begin{cases} 0 & * \neq 1 \\ \mathcal{E}_{\ell_1} + \mathcal{E}_{\ell_2} \subset \mathcal{E}_w & * = 1. \end{cases}$$

Proof of lemma. One has

$$\oplus_i H_1(\ell_i, \{0, p_i\}; E, \nabla) \to H_1(W, \delta_W \cup \{0\}; E, \nabla)$$

and of course the assertion of the lemma is that this coincides with $\mathcal{E}_{\ell_1} \oplus \mathcal{E}_{\ell_2} \to \mathcal{E}_{\ell_1} + \mathcal{E}_{\ell_2}$. To check this, by (1.5) one is reduced to the case $E = L \otimes M$ where L has rank 1 and M has regular singular points.

If W does not contain a Stokes line for L then $\mathcal{E}_{\ell_1} = \mathcal{E}_{\ell_2} = \mathcal{E}_{\ell_1} + \mathcal{E}_{\ell_2}$, and the argument is exactly as in lemma 1.2.

Suppose W contains a Stokes line for L. Then (say) $\mathcal{E}_{\ell_1} = \mathcal{E}_w$ and $\mathcal{E}_{\ell_2} = (0)$. Let $\mathcal{C}_*(W)$ be the complex of chains calculating the desired homology, and let $\mathcal{C}_*(W \setminus \{0\}) \subset \mathcal{C}_*(W)$ be the chains not meeting 0. As in the previous lemma, $\mathcal{C}_*(W \setminus \{0\})$ is acyclic. We claim the map

$$\mathcal{C}_*(\ell_1) \to \mathcal{C}_*(W)/\mathcal{C}_*(W \setminus \{0\})$$

is a quasi-isomorphism. If we choose an angular coordinate θ such that

$$\ell_1 : \theta = 0; \quad \text{Stokes} : \theta = a > 0; \quad \ell_2 : \theta = b > a,$$

then rotation $re^{i\theta} \mapsto re^{(1-t)i\theta}$ provides a homotopy contraction of the inclusion of $\ell_1 \subset W$. This homotopy contraction preserves the condition of rapid decay, proving the lemma. □

Let $\pi_d : \Delta \to \Delta$ be the ramified cover of degree d obtained by taking the d-th root of a parameter at 0. By the theory of formal connections [4], one has, for suitable d, a decomposition as in (1.5) for the formal completion of the pullback $\widehat{\pi_d^* E} \cong \oplus_i L_i \otimes M_i$. Let m_i be the degree of the pole of the connection on L_i when we identify $L_i \cong \widehat{\mathcal{O}}$, i.e. $\nabla_{L_i}(1) = g_i(z)dz$ for a local parameter z, and m_i is the order of pole of g_i.

Lemma 1.4. *We have*

$$\dim H_p(\Delta, \delta \cup \{0\}; E, \nabla) = \begin{cases} 0 & p \neq 1 \\ \frac{1}{d} \sum_{m_i \geq 2} (m_i - 1) \dim(M_i) & p = 1. \end{cases}$$

Proof of lemma. Assume first that we have a decomposition of the type (1.5) on $\widehat{E}$ itself, i.e. that no pullback π_d^* is necessary. We write Δ as a union of closed sectors $W_0, \ldots, W_{N-1}$ where W_i has radial boundary lines ℓ_i and ℓ_{i+1}. We assume each W_i has at most one Stokes line. Using excision together with the previous lemmas we get

$$\begin{aligned} (1.6) \quad 0 \to H_2(\Delta, \delta \cup \{0\}; E, \nabla) &\to \oplus_{i=0}^{N-1} H_1(\ell_i, \{p_i, 0\}; E, \nabla) \\ &\xrightarrow{\nu} \oplus_{i=0}^{N-1} H_1(W_i, \delta_{W_i} \cup \{0\}; E, \nabla) \to H_1(\Delta, \delta \cup \{0\}; E, \nabla) \to 0. \end{aligned}$$

By lemma 1.3, the map ν above is given by

$$\nu(e_0, \ldots, e_{N-1}) = (e_0 - e_1, e_1 - e_2, \ldots, e_{N-1} - e_0).$$

An element in the kernel of ν is thus a section e of $\mathcal{E}|_{\Delta-\{0\}}$ which has rapid decay along each ℓ_i. Since each W_i contains at most one Stokes line, such an e would necessarily have rapid decay on every sector and thus would be trivial. This proves vanishing for $H_2(\Delta,\delta;E,\nabla)$. Finally, to compute the dimension of H_1, note that if L_i has a connection with pole of order m_i, then it has a horizontal section of the form e^f, where f has a pole of order m_i-1. (The connection is $1\mapsto df$.) Suppose $f=az^{1-m_i}+\ldots$. Stokes lines for this factor are radial lines where az^{1-m_i} is pure imaginary. Thus, there are $2(m_i-1)$ Stokes lines for this factor. Consider one of the Stokes lines, and suppose it lies in W_k. If the real part of az^{1-m_i} changes from negative to positive as we rotate clockwise through this line, say we are in case $+$, otherwise we are in case $-$. We have

$$\dim(\mathcal{E}_{\ell_k}+\mathcal{E}_{\ell_{k+1}})-\dim\mathcal{E}_{\ell_k}=\begin{cases}0 & \text{case } +\\ \dim(M_i) & \text{case } -,\end{cases} \tag{1.7}$$

since the two cases alternate, we get a contribution of $(m_i-1)\dim(M_i)$. If $m_i\le 1$ there are no rapidly decaying sections, so that case can be ignored. Summing over i with $m_i\ge 2$ gives the desired result.

Finally, we must consider the general case when the decomposition (1.5) is only available on $\widehat{\pi_d^*E}$ for some $d\ge 2$. By a trace argument, vanishing of the homology upstairs, i.e. for $\widehat{\pi_d^*E}$, in degrees $\neq 1$ implies vanishing downstairs. Since $\pi_d:\Delta\setminus\{0\}\to\Delta\setminus\{0\}$ is unramified, an Euler characteristic argument (or, more concretely, just cutting into small sectors over which the covering splits) shows that the Euler characteristic multiplies by d under pullback, proving the lemma. □

In particular, we have now completed the proof of theorem 1.1. □

2. de Rham Cohomology

In this section, using differential forms, we construct the dual sequence to the homology sequence from theorem 1.1. (More precisely, we continue to work with E,∇, so the sequence we construct will be dual to the homology sequence with coefficients in $E^\vee,\nabla^\vee$). Consider the diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E(*D) & \longrightarrow & j_*E_U & \longrightarrow & j_*E_U/E(*D) & \longrightarrow & 0\\ & & \Big\downarrow{\scriptstyle\nabla_{\mathrm{mero}}} & & \Big\downarrow{\scriptstyle\nabla_{\mathrm{an}}} & & \Big\downarrow{\scriptstyle\nabla_{\mathrm{an/mero}}} & & \\ 0 & \longrightarrow & E(*D)\otimes\omega & \longrightarrow & j_*E_U\otimes\omega & \longrightarrow & (j_*E_U/E(*D))\otimes\omega & \longrightarrow & 0. \end{array} \tag{2.1}$$

A result of Malgrange [6] is that $\nabla_{\mathrm{an/mero}}$ is surjective. Define $N:=\oplus_i N_i=\ker(\nabla_{\mathrm{an/mero}})$. Since none of these sheaves has higher cohomology (by assumption $D\neq\emptyset$) we get a 5-term exact sequence by taking global sections and applying the serpent lemma:

$$(2.2)\quad 0 \to H^0_{DR}(U;E,\nabla) \to H^0(U,\mathcal{E}) \to N \to H^1_{DR}(U;E,\nabla) \to H^1(U,\mathcal{E}) \to 0.$$

Theorem 2.1. *Integration of forms over chains defines a perfect pairing between the exact sequence* (2.2) *and the exact sequence from theorem 1.1:*

$$(2.3)\quad 0 \to H_1(U,\mathcal{E}^\vee) \to H_1(X,D;E^\vee,\nabla^\vee) \to \oplus_i H_1(\Delta_i,\delta_i \cup \{x_i\};E^\vee,\nabla^\vee) \to H_0(U,\mathcal{E}^\vee) \to H_0(X,D;E^\vee,\nabla^\vee) \to 0.$$

Proof. To establish the existence of a pairing, note that if $c \otimes \epsilon^\vee$ is a rapidly decaying chain and η is a form of the same degree with moderate growth, then elementary estimates show $\int_c \langle \epsilon^\vee, \eta\rangle$ is well defined. Suppose $c : \Delta^n \to X$ and write $\Delta^n = \lim_{t\to 0} \Delta^n_t$ where Δ^n_t denotes $\Delta^n \setminus$ tubular neighborhood of radius t around $\partial\Delta^n$. Let $c_t = c|_{\Delta^n_t}$ and suppose $\eta = d\tau$ where τ has moderate growth also. Then

$$(2.4)\qquad \int_c \langle \epsilon^\vee, \eta\rangle = \lim_{t\to 0}\int_{c_t}\langle \epsilon^\vee, d\tau\rangle = \lim_{t\to 0}\int_{\partial c_t}\langle \epsilon^\vee,\tau\rangle = \int_{\partial c}\langle \epsilon^\vee,\tau\rangle.$$

Note ∂c may include simplices mapping to D. Our definition (0.5) of $\mathcal{C}_*(X,D;E^\vee,\nabla^\vee)$ factors these chains out. Thus, we do get a pairing of complexes.

Of course, chains away from D integrate with forms with possible essential singularities on D. To complete the description of the pairing, we must indicate a pairing

$$(2.5)\qquad (\ ,\ ) : N_i \times H_1(\Delta_i,\delta_i \cup \{x_i\};E^\vee,\nabla^\vee) \to \mathbb{C}.$$

To simplify notation we will drop the subscript i and take $x_i = 0$. An element in H_1 can be represented in the form $\epsilon^\vee \otimes c$ where c is a radial path. Let $c \cap \delta = \{p\}$. Given $n \in N$, choose a sector W containing c on which $\mathcal{E}$ has a basis ϵ_i. By assumption, we can represent $n = \sum a_i\epsilon_i$ with a_i analytic on the open sector, such that

$$(2.6)\qquad \nabla(\sum a_i\epsilon_i) = \sum \epsilon_i \otimes da_i = \sum e_i \otimes \eta_i$$

where e_i from a basis of E in a neighborhood of 0 and η_i are meromorphic 1-forms at 0. then by definition

$$(2.7)\qquad (\epsilon^\vee \otimes c, n) := \int_c \sum_i \langle \epsilon^\vee, e_i\rangle \eta_i - \sum_i \langle \epsilon^\vee, \epsilon_i\rangle a_i(p).$$

The pairing is taken to be trivial on chains which do not contain 0. If s is a 2-chain bounding two radial segments c and c' and a path along δ from

p to p'. Then Cauchy's theorem (together with a limiting argument at 0) gives

$$(2.8)\quad 0=\int_c\sum_i\langle\epsilon^\vee,e_i\rangle\eta_i-\int_{c'}\sum_i\langle\epsilon^\vee,e_i\rangle\eta_i+\int_p^{p'}\sum_i\langle\epsilon^\vee,\epsilon_i\rangle da_i$$
$$=(\epsilon^\vee\otimes c,n)-(\epsilon^\vee\otimes c',n).$$

Similar arguments show the pairing independent of the choice of the radius of the disk. Also, if $\sum a_i\epsilon_i=\sum b_ie_i$ with b_i meromorphic at 0, then

$$\begin{aligned}(2.9)\qquad (\epsilon^\vee\otimes c,n)&=\int_c\sum\langle\epsilon^\vee,e_i\rangle\eta_i-\sum_i\langle\epsilon^\vee,\epsilon_i\rangle a_i(p)\\&=\int_c d\langle\epsilon^\vee,\sum b_ie_i\rangle-\sum_i\langle\epsilon^\vee,\epsilon_i\rangle a_i(p)\\&=\langle\epsilon^\vee,\sum b_ie_i\rangle(p)-\sum_i\langle\epsilon^\vee,\epsilon_i\rangle a_i(p)=0.\end{aligned}$$

It follows that the pairing is well defined.

Lemma 2.2. *The diagrams*

$$\begin{array}{ccc}H_1(X,D;E^\vee,\nabla^\vee) & \to & \oplus H_1(\Delta_i,\delta_i;E^\vee,\nabla^\vee)\\ \times & & \times\\ H^1(X\setminus D;E,\nabla) & \leftarrow & \oplus N_i\\ & \searrow\quad\mathbb{C}\quad\swarrow & \end{array}$$

and

$$\begin{array}{ccc}\oplus H_1(\Delta_i,\delta_i;E^\vee,\nabla^\vee) & \to & H_0(U,\mathcal{E}^\vee)\\ \times & & \times\\ \oplus N_i & \leftarrow & H^0(U,\mathcal{E})\\ & \searrow\quad\mathbb{C}\quad\swarrow & \end{array}$$

commute.

Proof of lemma. Consider the top square. The top arrow is excision, replacing a chain with the part of it lying in the disks Δ_i. The bottom arrow maps an n as above in some N_i to $\sum e_j\otimes\eta_j=\sum\epsilon_j\otimes da_j$. Along c outside the disks $\sum e_j\otimes\eta_j$ is exact; its integral along the chain is a sum of terms of the form $\sum_i\langle\epsilon^\vee,\epsilon_j\rangle a_j(p_i)$ where $p_i\in c\cap\delta_i$. For the part of the chain inside the Δ_i of course we must take $\int_{c\cap\Delta_i}\langle\epsilon^\vee,e_j\rangle\eta_j$. Combining these terms with appropriate signs yields the desired compatibility.

For the bottom square, the top arrow associates to a relative chain on Δ_i its boundary on $\delta_i\subset U$. The bottom arrow associates to a horizontal section ϵ on U the corresponding element in N. Note here the a_j will be constant so in the pairing with N only the term $-\sum\langle\epsilon^\vee, \epsilon_j\rangle a_j(p)$ survives. The assertion of the lemma follows. $\square$

Returning to the proof of the theorem, we see it reduces to a purely local statement for a connection on a disk. In the following lemma, we modify notation, writing N to denote the corresponding group for a connection on a disk Δ with a meromorphic singularity at 0.

Lemma 2.3. *The pairing*

$$(\ , \ ) : N \times H_1(\Delta, \delta; E^\vee, \nabla^\vee) \to \mathbb{C}$$

is nondegenerate on the left, i.e. $(\epsilon^\vee \otimes c, n) = 0$ *for all relative* 1*-cycles implies* $n = 0$.

Proof of lemma. We work in a sector and we suppose the basis ϵ_i taken in the usual way compatible (in the sector) with the decomposition into a direct sum of rank 1 irregular connections tensor regular singular point connections. Let $\epsilon_i^\vee$ be the dual basis.

Fix an i and suppose first ϵ_i and $\epsilon_i^\vee$ both have moderate growth. We claim a_i has moderate growth. For this it suffices to show da_i has moderate growth. But

$$da_i = \langle\nabla(n), \epsilon_i^\vee\rangle = \sum_j \langle e_j, \epsilon_i^\vee\rangle\eta_j. \tag{2.10}$$

This has moderate growth because, $e_j, \epsilon_i^\vee$, and η_j all do.

Now assume $(\epsilon^\vee \otimes c, n) = 0$ for all $\epsilon^\vee \otimes c \in H_1$. Fix an i and assume $\epsilon_i^\vee$ is rapidly decreasing in our sector. Let c be a radius in the sector with endpoint p. We can find (cf. [4], chap. IV, p.53-56) a basis t_i of E on the sector with moderate growth and such that $t_i = \psi_i\epsilon_i$, so $t_i^\vee = \psi_i^{-1}\epsilon_i^\vee$.

We are interested in the growth of $a_i\epsilon_i$ along c. We have

(2.11)
$$a_i(p)\epsilon_i(p) = \Big(\int_c \sum_j \langle\epsilon_i^\vee, e_j\rangle\eta_j\Big)\epsilon_i(p) = \Big(\int_c \psi_i \sum_j \langle t_i^\vee, e_j\rangle\eta_j\Big)\psi_i(p)^{-1}t_i(p).$$

Asymptotically, taking y the parameter along c, $\psi_i(y) \sim \exp(-ky^{-N})$ as $y \to 0$ for some $k > 0$ and some $N \geq 1$. We need to know the integral

$$\exp(kp^{-N}) \int_0^p y^{-M} \exp(-ky^{-N})dy \tag{2.12}$$

has moderate growth as $p \to 0$. Changing variables, so $x = y^{-1}, q = p^{-1}, u = x - q$, this becomes

$$(2.13)\quad \int_0^\infty (u+q)^{M-2} \exp(q^N - (u+q)^N) du$$
$$= \int_0^\infty (u+q)^{M-2} \exp(-u^N - qf(u,q)) du,$$

where f is a sum of monomials in q and u with positive coefficients. Clearly this has at worst polynomial growth as $q \to \infty$ as desired.

Finally, assume $\epsilon_i^\vee$ is rapidly increasing and ϵ_i is rapidly decreasing. We have as above

$$(2.14)\qquad \sum_i e_j \otimes \eta_j = \sum_j \epsilon_j \otimes da_j = \sum_j \psi_j^{-1} t_j \otimes da_j.$$

In particular, $\psi_i^{-1} da_i$ has moderate growth. This implies $a_i\epsilon_i = a_i\psi_i^{-1}t_i$ has moderate growth as well. Indeed, changing notation, this amounts to the assertion that if g is rapidly decreasing and $g\frac{df}{dz}$ has moderate growth, then gf has moderate growth. Fix a point p_0 with $0 < p < p_0$. the mean value theorem says there exists an r with $p \leq r \leq p_0$ such that

$$g(p)f(p) = g(p)(f(p_0) + (p - p_0)f'(r)).$$

Suppose $|f'(q)g(q)| << q^{-N}$. We get

$$|g(p)f(p)| << |g(p)f'(r)| \leq |g(r)f'(r)| << r^{-N} \leq p^{-N}$$

proving moderate growth.

We conclude that our representation for n has moderate growth, and hence it is zero in N. It follows that the pairing $N \times H_1 \to \mathbb{C}$ is nondegenerate on the left. $\square$

Returning to the global situation, we have now

$$\dim N_i \leq \dim H_1(\Delta_i, \delta_i; E^\vee, \nabla^\vee),$$

and to finish the proof of the theorem, it will suffice to show these dimensions are equal.

Lemma 2.4. *With notation as above,* $\dim N_i = \dim H_1(\Delta_i, \delta_i; E^\vee, \nabla^\vee)$.

Proof of lemma. It will suffice to compute the difference of the two Euler characteristics

$$(2.15)\qquad \chi(U, \mathcal{E}) - \chi_{DR}(U; E, \nabla).$$

It is straightforward to show this difference is invariant if U is replaced by a smaller Zariski open set, and that the Euler characteristics are multiplied by the degree in a finite étale covering $V \to U$. Using lemma 1.4, we reduce to the case where formally locally at each $x_i \in D$ we have

$E \otimes \widehat{K}_{x_i} \cong \oplus_j L_{ij} \otimes M_{ij}$ with L_{ij} rank 1 and M_{ij} at worst regular singular. (Here $\widehat{K}_{x_i}$ is the Laurent power series field at x_i). Let m_{ij} be the degree of the pole for the connection on L_{ij}. Then one can find coherent sheaves

$$F_2 \subset F_1 \subset E(*D)$$

such that

$$F_1/F_2 \cong \oplus_{ij} M_{ij}/M_{ij}(-m_{ij}x_i),$$
$$E^{\nabla} \subset H^0(F_2); \nabla(F_2) \subset F_1 \otimes \omega,$$
$$H^0(F_1 \otimes \omega) \twoheadrightarrow H^1_{DR}(U; E, \nabla).$$

It follows that, writing $g = \mathrm{genus}(X)$,

$$\text{(2.16)} \quad \chi_{DR}(U; E, \nabla) = \chi(F_2) - \chi(F_1 \otimes \omega) = -\mathrm{rk}(E)(2g-2) - \sum_{ij} m_{ij} \dim(M_{ij}).$$

Since

$$\text{(2.17)} \qquad \chi(U, \mathcal{E}) = -\mathrm{rk}(E)(2g - 2 + n),$$

(which is proven algebraically as above, replacing ∇ by the regular connection associated to $\mathcal{E}$) it follows that

$$\chi_{DR}(U; E, \nabla) - \chi(U, \mathcal{E}) = -\sum_{ij} (m_{ij} - 1) \dim(M_{ij}).$$

Referring to lemma 1.4, we see that this is the desired formula. □

This completes the proof of the theorem. □

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