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Diophantine inequalities with power sums

par Amedeo SCREMIN

Résumé. On appelle somme de puissances toute suite $\alpha : \mathbb{N} \to \mathbb{C}$ de nombres complexes de la forme

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n,$$

où les $b_i \in \overline{\mathbb{Q}}$ et les $c_i \in \mathbb{Z}$ sont fixés. Soit $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ un polynôme unitaire, absolument irréductible, de degré au moins 2 en y. On démontre que les solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ de l'inégalité

$$|F(\alpha(n), y)| < \left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot |\alpha(n)|^{-\varepsilon}$$

sont paramétrées par un nombre fini de sommes de puissances. Par conséquent, on déduit la finitude des solutions de l'équation diophantienne

$$F(\alpha(n), y) = f(n),$$

où $f \in \mathbb{Z}[x]$ est un polynôme non constant et α est une somme de puissances non constante.

Abstract. The ring of power sums is formed by complex functions on $\mathbb{N}$ of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n,$$

for some $b_i \in \overline{\mathbb{Q}}$ and $c_i \in \mathbb{Z}$. Let $F(x, y) \in \overline{\mathbb{Q}}[x, y]$ be absolutely irreducible, monic and of degree at least 2 in y. We consider Diophantine inequalities of the form

$$|F(\alpha(n), y)| < \left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot |\alpha(n)|^{-\varepsilon}$$

and show that all the solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ have y parametrized by some power sums in a finite set. As a consequence, we prove that the equation

$$F(\alpha(n), y) = f(n),$$

with $f \in \mathbb{Z}[x]$ not constant, F monic in y and α not constant, has only finitely many solutions.

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1. Introduction

The present paper deals with diophantine equations and inequalities involving certain *power sums*, i.e. functions of $n \in \mathbb{N}$ of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n \ , \tag{1}$$

with $c_1 > c_2 > \ldots > c_h > 0$, where the b_i, called the *coefficients* of $\alpha(n)$, are (nonzero) algebraic numbers and the c_i, called the *roots* of $\alpha(n)$, are distinct integers or rationals. A power sum is *non-degenerate* if no quotient of two distinct roots is a root of unity. It is well known that such functions, even allowing the b_i to be polynomials in n and the c_i to be algebraic numbers, satisfy linear recurrence relations. Since long ago, a number of results concerning diophantine equations and inequalities with power sums have been proved. Among the recent ones, we may mention, for instance, the results by Kiss [9] who proved, under some assumptions on the absolute values of the roots of $\alpha(n)$, that the inequality $|sx^q - \alpha(n)| > e^{cn}$, where $\alpha(n)$ is a non-degenerate power sum with algebraic roots and polynomial coefficients, holds for integers s, $x > 1$ and q, provided that n and q are large enough. Shorey and Stewart [14] proved that for any fixed $\delta > 0$ the inequality $|sx^q - \alpha(n)| > |c_1|^{n(1-\delta)}$, where $\alpha(n)$ is non-degenerate with algebraic roots and constant coefficients, holds for all the non-zero integers s, x, for $n > 0$, and for every non-zero integer $q > q_o(\alpha, P)$, where P is the greatest prime factor of s, assuming that $sx^q \neq b_1 c_1^n$ and that in $\alpha(n)$ there is a root with largest absolute value. This result was proved using estimates for linear forms in logarithms due to Baker (see [1]). Pethö [10] proved for non-degenerate power sums with $h = 2$ and coprime coefficients that if $\alpha(n) = sx^q$ holds for integers $x \neq 0$, $q \geq 2$ and $n > 0$, then $\max\{|x|, q, n\}$ is bounded by an effectively computable number depending on the greatest prime divisor of s. Recently Corvaja and Zannier [2] have found new results concerning the inequality $|\alpha(n) - y^d| \ll |\alpha(n)|^\rho$, where $\alpha(n)$ has positive integral roots and rational coefficients, $d \geq 2$ and $\rho < 1 - 1/d$. Using the Schmidt Subspace Theorem (see [12]) they proved that if this inequality has infinitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$, then all the solutions, but finitely many, have y parametrized by some power sums in a finite set; also, the numbers n such that (n, y) is a solution, except finitely many, form a finite union of arithmetical progressions. As a consequence, for every $d \geq 2$ the equation $\alpha(n) = y^d$ has only finitely many solutions, if we suppose that $\alpha(n)$ has positive integral roots and that two roots with largest absolute value are coprime, apart from trivial cases, which are easy to classify. In [3], under some assumptions on the size of the roots of $\alpha(n)$ and allowing the coefficients and the roots of $\alpha(n)$ to be algebraic, they extended this result to the more general equation $F(\alpha(n), y) = 0$. This paper will not be concerned with quantitative aspects, though the methods allow to estimate the number of relevant solutions. In the context of the paper by Corvaja and Zannier ([3]), some extimates have been obtained by Fuchs [7], using a quantitative version of the Subspace Theorem due to Evertse (see [6]).

In this paper we first study lower bounds for the quantity $|F(\alpha(n),y)|$, and in particular the inequality $|F(\alpha(n),y)| < |\frac{\partial F}{\partial y}(\alpha(n),y)| \cdot |\alpha(n)|^{-\varepsilon}$ for power sums with integral roots and algebraic coefficients, where $F(x,y)$ is an absolutely irreducible polynomial monic in y. We shall obtain (Theorem 3.1) that all the solutions $(n,y) \in \mathbb{N} \times \mathbb{Z}$ have y parametrized by some power sums in a finite set. This conclusion is in a sense best possible, since the same result doesn't hold for $\varepsilon < 0$. In fact, suppose that $F(\alpha(n),y)$ has a real zero y_n for all sufficiently large n. Setting $y(n)$ to be the nearest integer to y_n, we have (see [4])

$$|F(\alpha(n),y(n)| = |y(n)-y_n||\frac{\partial F}{\partial y}(\alpha(n),\xi)| < |\frac{\partial F}{\partial y}(\alpha(n),\xi)| \ll |\frac{\partial F}{\partial y}(\alpha(n),y(n))|,$$

where $y(n) \le \xi \le y_n$ (or $y_n \le \xi \le y(n)$).

Our proof shall use a result concerning the inequality $|\alpha(n)-y| < e^{-n\varepsilon}$ derived by Corvaja and Zannier [2, Lemma 2] from Schmidt Subspace Theorem. From Theorem 3.1 follows (Corollary 3.2) the generalization of the result in [2, Theorem 3] to the inequality $|F(\alpha(n),y)| < |\alpha(n)|^{1-\frac{1}{d}-\varepsilon}$, under some assumptions on the Puiseux expansion at infinity of y as function of x under the relation $F(x,y)=0$. As a simple application (Corollary 3.3) we shall deduce the finiteness of the solutions of the equation $F(\alpha(n),y) = f(n)$, under the assumption that $f(n)$ is a non constant polynomial and that $\alpha(n)$ is not constant. This gives a generalization of the results in [2] and [3].

2. Notation

In the present paper we will denote by $\Sigma_{\mathbb{Q}}$ and $\Sigma_{\mathbb{Z}}$ the rings of functions on $\mathbb{N}$ of the form $\alpha(n) = b_1c_1^n + b_2c_2^n + \ldots + b_hc_h^n$, where the distinct roots $c_i \neq 0$ are in $\mathbb{Q}$ or in $\mathbb{Z}$ respectively, and the coefficients $b_i \in \mathbb{Q}^\star$. If $\mathbb{K} \subset \mathbb{C}$ is a number field, we will denote by $\mathbb{K}\Sigma_{\mathbb{Q}}$ and $\mathbb{K}\Sigma_{\mathbb{Z}}$ the ring of power sums with coefficients in $\mathbb{K}$.

The subrings of power sums with only positive roots will be denoted by $\mathbb{K}\Sigma_{\mathbb{Q}}^+$ and $\mathbb{K}\Sigma_{\mathbb{Z}}^+$. Working in this domain causes no loss of generality: the assumption of positivity of the roots may usually be achieved by writing $2n+r$ instead of n, and considering the cases of $r=0,1$ separately.

Note that every constant power sum, i.e. a power sum with only one root $c_1 = 1$, belongs to $\Sigma_{\mathbb{Z}}^+$. Power sums will be denoted by Greek letters.

3. Statements

Theorem 3.1. *Let $F\in\overline{\mathbb{Q}}[x,y]$ be absolutely irreducible, monic and of degree $d\geq 2$ in y; let $\alpha(n)\in\overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$, and let $\varepsilon>0$ be fixed. Then there exists a finite set of power sums $\{\beta_1(n),\dots,\beta_s(n)\}\subset\Sigma_{\mathbb{Z}}^{+}$ such that every solution $(n,y)\in\mathbb{N}\times\mathbb{Z}$ of the inequality*

$$|F(\alpha(n),y)|<\left|\frac{\partial F}{\partial y}(\alpha(n),y)\right|\cdot|\alpha(n)|^{-\varepsilon} \tag{2}$$

satisfies $y=\beta_i(n)$, for a certain $i=1,\dots,s$.

The set $\{\beta_1(n),\dots,\beta_s(n)\}$ contains at most d^2 non constant power sums. Moreover, the set of natural numbers n such that (n,y) is a solution of (2) is the union of a finite set and a finite number of arithmetic progressions.

For the formulation of Corollary 3.2 we need the following.

Definition. Let $F(x,y)\in\overline{\mathbb{Q}}[x,y]$ be monic in y and of degree $d\geq 2$ in y. Let $F(x,y)=(y-\varphi_1(x))\cdot\ldots\cdot(y-\varphi_d(x))$ be the factorization of $F(x,y)$ in the ring of Puiseux series in x at infinity. Here, for each $j=1,\dots,d$, $\varphi_j(x)=\sum_{i=-k_j}^{+\infty}a_{ij}\,x^{-i/e_j}$, with $a_{-k_j j}\neq 0$ and for a real determination of x^{1/e_j}, is an expansion at infinity of y as function of x.

In the present paper we will call the polynomial $F(x,y)$ *"regular"* if for every $j,l=1,\dots,d$, with $j\neq l$, we have $k_j/e_j\neq k_l/e_l$ or $a_{-k_j j}\neq a_{-k_l l}$.

Corollary 3.2. *Let $F(x,y)\in\overline{\mathbb{Q}}[x,y]$ be monic in y, absolutely irreducible, regular, of degree $d\geq 2$ in y. Let $\alpha(n)\in\overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$; let $\varepsilon>0$ and $c>0$ be fixed. Then there exists a finite set of power sums $\{\beta_1(n),\dots,\beta_s(n)\}\subset\Sigma_{\mathbb{Z}}^{+}$ such that every solution $(n,y)\in\mathbb{N}\times\mathbb{Z}$ of the inequality*

$$|F(\alpha(n),y)|<c\cdot|\alpha(n)|^{1-\frac{1}{d}-\varepsilon} \tag{3}$$

satisfies $y=\beta_i(n)$ for a certain $i=1,\dots,s$.

The set $\{\beta_1(n),\dots,\beta_s(n)\}$ contains at most d^2 non constant power sums. Moreover, the natural numbers n such that (n,y) is a solution of (3), except finitely many, make up a finite union of arithmetical progressions.

Corollary 3.3. *Let $F(x,y)\in\overline{\mathbb{Q}}[x,y]$ be monic in y, absolutely irreducible and of degree $d\geq 2$ in y; let $f(n)\in\mathbb{Z}[x]$ be a non constant polynomial; let $\alpha(n)$ be a non constant power sum with integral roots and algebraic coefficients. Then the equation*

$$F(\alpha(n),y)=f(n) \tag{4}$$

has only finitely many solutions $(n,y)\in\mathbb{N}\times\mathbb{Z}$.

4. Auxiliary results

The following Lemma 4.1, proved in a more general version by Corvaja and Zannier (see [2, Lemma 2]) using a version of the Subspace Theorem due to H.P. Schlickewei (see [11], [12, Theorem 1, p. 178]), plays a crucial role throughout the paper, since it contains the fundamental information to prove Theorem 3.1.

Lemma 4.1. *Let* $\tau(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}^{+}$, *and let* $\varepsilon > 0$ *be fixed. Then there exists a power sum* $\beta(n) \in \Sigma_{\mathbb{Z}}^{+}$ *such that for all but finitely many solutions* $(n, y) \in \mathbb{N} \times \mathbb{Z}$ *of the inequality*

$$|\tau(n) - y| < e^{-n\varepsilon}, \tag{5}$$

we have $y = \beta(n)$.

Moreover, the roots of $\beta(n)$ *are in the set of the roots of* $\tau(n)$.

For the proof of Theorem 3.1 we need also some standard results from the theory of algebraic functions fields, namely the theory of Puiseux expansions. We recall here a simple version of the Puiseux Theorem concerning the Puiseux expansions at the infinity for the polynomials of $\overline{\mathbb{Q}}(x)[y]$. More general versions can be found in [5] and [8].

Theorem 4.2 (Puiseux Theorem)**.** *Let* $F(x, y) \in \overline{\mathbb{Q}}(x)[y]$ *be an absolutely irreducible polynomial, monic and of degree* d *in* y. *Then for* $i = 1, \dots, d$ *there exist* $e_i \in \mathbb{N}$, $1 \le e_i \le d$, *and Laurent series in* x^{-1/e_i}

$$\varphi_i(x) = \sum_{k=v_i}^{+\infty} a_{ik}\, x^{-k/e_i}, \quad i = 1, \dots, d$$

with $v_i \le 0$, *such that*

$$F(x, y) = \prod_{1=1,\dots,d} (y - \varphi_i(x)).$$

The Laurent series $\varphi_1(x), \dots, \varphi_d(x)$ *are convergent for* $|x|$ *large enough, and the coefficients* a_{ij} *are elements of a finite field extension* $\mathbb{K}$ *of* $\mathbb{Q}$.

The Laurent series $\varphi_1(x), \dots, \varphi_d(x)$ coming from the Puiseux Theorem are called *Puiseux series* of the polynomial $F(x, y)$.

5. Proofs

Proof of Theorem 3.1. Plainly, we need to consider only the case that (2) has infinitely many solutions. We shall consider solutions with n larger than a certain constant N, since the finitely many solutions with $n \le N$ can be considered as constant power sums. Finally, we can suppose $\alpha(n)$ not constant.

Let $F(x, y) = (y - \varphi_1(x)) \cdot \ldots \cdot (y - \varphi_d(x))$, where

$$\varphi_j(x) = \sum_{i=-k_j}^{+\infty} a_{ij}\, x^{-i/e_j}, \text{ with } a_{-k_j j} \neq 0 \text{ and } 1 \leq e_j \leq d \text{ for } j = 1, \dots, d,$$

are the series of the Puiseux expansion at infinity of y as function of x (see Theorem 4.2), i.e. $\varphi_j(x)$ are the solutions of the equation $F(x, y) = 0$ in the field of the Puiseux series.

Let us remark that by the Puiseux Theorem the series $\varphi_j(x)$ exist and the coefficients a_{ij} generate a finite field extension $\mathbb{K}$ of $\mathbb{Q}$.

We have $\frac{\partial F}{\partial y}(x, y) = \sum_{j=1}^{d} \frac{F(x,y)}{y-\varphi_j(x)}$, and so

$$F(x, y) = \frac{\partial F}{\partial y}(x, y)\Big(\sum_{j=1}^{d} \frac{1}{y - \varphi_j(x)}\Big)^{-1} \tag{6}$$

holds.

From (6) we obtain that for each solution (n, y) of (2) the inequality

$$\left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot \Big|\sum_{j=1}^{d} (y - \varphi_j(\alpha(n)))^{-1}\Big|^{-1} < \left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \cdot |\alpha(n)|^{-\varepsilon}$$

holds. By (2), we can assume $\left|\frac{\partial F}{\partial y}(\alpha(n), y)\right| \neq 0$. It follows that

$$\Big|\sum_{j=1}^{d} (y - \varphi_j(\alpha(n)))^{-1}\Big| > |\alpha(n)|^{\varepsilon}$$

holds, and so for all the solutions of (2) we have

$$\sum_{j=1}^{d} \Big|y - \varphi_j(\alpha(n))\Big|^{-1} > |\alpha(n)|^{\varepsilon}.$$

Let $\varepsilon_1 = \dfrac{\varepsilon}{2}$. For n large enough the inequality $\sum_{j=1}^{d} |y - \varphi_j(\alpha(n))|^{-1} > d \cdot |\alpha(n)|^{\varepsilon_1}$ holds, and so for a certain $j = 1, \dots, d$ we have $|y - \varphi_j(\alpha(n))|^{-1} > |\alpha(n)|^{\varepsilon_1}$. This means that for every solution (n, y) of (2) with n large enough the inequality

$$|y - \varphi_j(\alpha(n))| < |\alpha(n)|^{-\varepsilon_1} \tag{7}$$

is satisfied for a certain $j = 1, \dots, d$, with j depending on n.

We shall prove that for given $j = 1, \dots, d$ there exists a finite set $\{\beta_1(n), \dots, \beta_t(n)\} \subset \Sigma_{\mathbb{Z}}$ such that every solution (n, y) of (7) has $y = \beta_i(n)$ for a certain $i = 1, \dots, t$.

Once we prove this, the theorem will follow.

Define a partition $\{M_1, \dots, M_d\}$ of the solutions (n, y) of (2) by prescribing that for every $(n, y) \in M_i$ we have

$$|y - \varphi_i(\alpha(n))| = \min_{1 \leq j \leq d} \{|y - \varphi_j(\alpha(n))|\}.$$

We can consider separately the solutions in each subset M_i. It will suffice to deal with $i = 1$.

Let us write

$$(8)\quad \varphi_1(x) = \sum_{i=-k}^{+\infty} a_i x^{-i/e_1} = a_{-k}x^{k/e_1} + \ldots + a_{-1}x^{1/e_1} + a_0 + a_1 x^{-1/e_1} + \ldots,$$

for a real determination of x^{1/e_1}, where $k = k_1$ and $a_i = a_{i,1}$ for every $i \geq -k$.

Let $\alpha(n) = \sum_{j=1}^{h} b_j c_j^n$, with $c_j \in \mathbb{Z}$, $c_j \neq 1$ for some j and $b_j \in \overline{\mathbb{Q}}$ $\forall\, j = 1, \ldots, h$. We can suppose $c_1 > c_2 > \ldots > c_h > 0$.

For n large enough the series $\varphi_1(\alpha(n))$ converges, so we can write

$$(9)\qquad \varphi_1(\alpha(n)) = \sum_{i=-k}^{0} a_i \alpha(n)^{-i/e_1} + O(\alpha(n)^{-1/e_1}).$$

Choosing $\varepsilon_2 > 0$ smaller than ε_1 and $1/e_1$, for n large enough each solution of $|y - \sum_{i=-k}^{+\infty} a_i \alpha(n)^{-i/e_1}| < |\alpha(n)|^{-\varepsilon_1}$ satisfies

$$|y - \sum_{i=-k}^{0} a_i \alpha(n)^{-i/e_1}| < |\alpha(n)|^{-\varepsilon_2}.$$

Put

$$(10)\qquad \tilde{\varphi_1}(x) = \sum_{i=-k}^{0} a_i x^{-i/e_1}.$$

From now on we will consider the inequality

$$(11)\qquad |y - \tilde{\varphi_1}(\alpha(n))| < |\alpha(n)|^{-\varepsilon_2}$$

instead of $|y - \varphi_1(\alpha(n))| < |\alpha(n)|^{-\varepsilon_1}$.

We can write $\alpha(n) = b_1 c_1^n (1 + \sigma(n))$, with $\sigma(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}$, and $\sigma(n) = O((c_2/c_1)^n)$.

For every $l \in \mathbb{N}$ we have

$$(12)\qquad \alpha(n)^{l/e_1} = b_1^{l/e_1} (c_1^n)^{l/e_1} (1 + \sigma(n))^{l/e_1},$$

for a real determination (resp. real positive) of b_1^{l/e_1} (resp. c_1^{l/e_1}). We will fix this determination for the remaining part of the proof.

Expanding the function $t \mapsto (1+t)^{l/e_1}$ in Taylor series, we have for every $l \in \mathbb{N}$

$$(13)\qquad (1 + \sigma(n))^{l/e_1} = 1 + \sum_{j=1}^{m} B_{j,l}\, \sigma(n)^j + O(|\sigma(n)|^{m+1}),$$

where m is an integer to be chosen later and $B_{j,l}$, $j = 1, \dots, m$, $l \in \mathbb{N}$, are the Taylor coefficients $\binom{l/e_i}{j}$ of the function $t \mapsto (1+t)^{l/e_1}$.

From (12) and (13) we obtain

$$\alpha(n)^{l/e_1} = b_1^{l/e_1} c_1^{nl/e_1} \Big(1 + \sum_{j=1}^{m} B_{j,l}\ \sigma(n)^j\Big) + O\big(|\sigma(n)|^{m+1} \cdot c_1^{nl/e_1}\big). \tag{14}$$

Let us define, for every $l \in \mathbb{N}$,

$$\gamma_l(n) := \sum_{j=1}^{m} B_{j,l}\ \sigma(n)^j \ \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}^+.$$

Since (14) holds, we can write

$$\alpha(n)^{l/e_1} = b_1^{l/e_1} c_1^{nl/e_1}\ (1 + \gamma_l(n)) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nl/e_1}\big). \tag{15}$$

From (10) and (15) we obtain

$$\tilde{\varphi}_1(\alpha(n)) = \sum_{i=-k}^{0} \Big(a_i (b_1 c_1^n)^{-i/e_1} (1 + \gamma_{-i}(n))\Big) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big). \tag{16}$$

Let us write $n = n_1 e_1 + r$, with $0 \le r < e_1 \le d$. We can rewrite (16) as

$$\tilde{\varphi}_1(\alpha(n)) = \sum_{i=-k}^{0} \Big(a_i (b_1 c_1^r)^{-i/e_1} c_1^{-n_1 i} (1 + \gamma_{-i}(n))\Big) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big). \tag{17}$$

Since $\overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}^+$ is a ring, we see that

$$\tau(n) := \sum_{i=-k}^{0} \Big(a_i (b_1 c_1^r)^{-i/e_1} c_1^{-n_1 i} (1 + \gamma_{-i}(n))\Big)$$

is a power sum with rational positive roots and algebraic coefficients. Moreover, its roots lie in the multiplicative group generated by the real e_1-th roots (as determined above) of the roots of the power sum $\alpha(n)$.

We can write

$$\tilde{\varphi}_1(\alpha(n)) = \tau(n) + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big). \tag{18}$$

So we have

$$|y - \tilde{\varphi}_1(\alpha(n))| = |y - \tau(n)| + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big)\ ,$$

and from (11) we obtain

$$|y - \tau(n)| < |\alpha(n)|^{-\varepsilon_2} + O\big((c_2^n/c_1^n)^{m+1} \cdot c_1^{nk/e_1}\big)\ . \tag{19}$$

Let us notice that for a fixed m large enough $(c_2^n/c_1^n)^{m+1}c_1^{nk/e_1} < |\alpha(n)|^{-\varepsilon_2}$ holds for every n large enough. Choosing a suitable m large enough, every solution of (19) with n large enough is also a solution of

$$|y - \tau(n)| < 2|\alpha(n)|^{-\varepsilon_2}. \tag{20}$$

Choosing $\varepsilon_3 > 0$ small enough, $2|\alpha(n)|^{-\varepsilon_2} < e^{-n\varepsilon_3}$ holds for n large enough, since $|\alpha(n)| \longrightarrow +\infty$ for $n \to +\infty$ (we are supposing $\alpha(n)$ not constant).

Thus the inequality (20) implies

$$|y - \tau(n)| < e^{-n\varepsilon_3}. \tag{21}$$

Applying Lemma 4.1 we obtain that every solution of (21), with finitely many exceptions, has $y = \beta_1(n)$, where $\beta_1(n) \in \Sigma_{\mathbb{Z}}^+$. The roots of the power sum $\beta_1(n)$ are in the set of the roots of $\tau(n)$, and so in the multiplicative group generated by the real e_1-th roots of the roots of the power sum $\alpha(n)$.

Let us notice that the finitely many solutions (n, y) of (21) such that $y \neq \beta_1(n)$ can be considered as constant power sums $\beta_2(n), \dots, \beta_r(n) \in \Sigma_{\mathbb{Z}}^+$ with a single root 1.

This means that for $j = 1$ every solution (n, y) of (7) has $y = \beta_i(n)$ for a certain $i \in \{1, \dots, t\}$, where $\{\beta_1, \dots, \beta_t\} \subset \overline{\mathbb{Q}}\Sigma_{\mathbb{Z}}$, with $t \geq r$.

In a similar way this result can be obtained for $j = 2, \dots, d$ in (7). So we have that every solution of (2) has $y = \beta_i(n)$ for a certain $i \in \{1, \dots, s\}$, where $\{\beta_1, \dots, \beta_s\} \subset \Sigma_{\mathbb{Z}}^+$, with $s \geq t$.

Since each of the Puiseux series $\varphi_j(x)$, $j = 1, \dots, d$, gives rise to at most e_j non constant power sums (remember that we chose $0 \leq r < e_j$ in (17) and that $e_j \leq d$ for every $j = 1, \dots, d$), the set $\{\beta_1(n), \dots, \beta_s(n)\}$ contains at most d^2 non constant power sums.

Finally, we note that the roots of the power sums $\beta_1(n), \dots, \beta_s(n)$ are positive integers lying in the multiplicative group generated by the real e-th roots, with $1 \leq e \leq d$, of the roots of the power sum $\alpha(n)$.

This proves the Theorem. □

Proof of Corollary 3.2 As in the proof of Theorem 3.1, we shall consider only solutions (n, y) of (3) with n larger than a certain constant N, since the solutions with $n \leq N$ are finite in number and can be considered as constant power sums.

Let $F(x, y) = (y - \varphi_1(x)) \cdot \ldots \cdot (y - \varphi_d(x))$, where $\varphi_j(x) = \sum_{i=-k_j}^{+\infty} a_{ij}\, x^{-i/e_j}$, with $a_{-k_j j} \neq 0$ and $1 \leq e_j \leq d$ for $j = 1, \dots, d$, are the series of the Puiseux expansion at infinity of y as function of x.

Let $\varepsilon_1 > 0$ to be chosen later. In the proof of Theorem 3.1 we have shown that there exists a finite set of power sums with positive integral roots and rational coefficients $\{\beta_1(n),\dots,\beta_t(n)\}$ such that, for every $j=1,\dots,d$, every solution $(n,y)\in\mathbb{N}\times\mathbb{Z}$ of the inequality

$$|y-\varphi_j(\alpha(n))|<|\alpha(n)|^{-\varepsilon_1}$$

has $y=\beta_i(n)$ for a certain $i=1,\dots,t$. Moreover, the set $\{\beta_1(n),\dots,\beta_t(n)\}$ contains at most d non constant power sums.

Let us consider sets $M_1,\dots,M_d$ of pairs $(n,y)\in\mathbb{N}\times\mathbb{Z}$ such that for every $(n,y)\in M_i$ we have

$$|y-\varphi_i(\alpha(n))|=\min_{1\le j\le d}\{|y-\varphi_j(\alpha(n))|\}.$$

As before, we can consider separately each set, say M_1.

For every $i=2,\dots,d$, we have

$$|y-\varphi_i(\alpha(n))|\ge \frac{1}{2}\,|\varphi_i(\alpha(n))-\varphi_1(\alpha(n))|. \tag{22}$$

Since the polynomial F is regular, we can have either that $k_i/e_i\ne k_1/e_1$, $\forall\, i=2,\dots,d$, or that there exist some $i\in\{2,\dots,d\}$ such that $k_i/e_i=k_1/e_1$, but $a_{-k_i i}\ne a_{-k_1 1}$.

If $k_i/e_i\ne k_1/e_1$ $\forall\, i=2,\dots,d$, for n large enough we have

$$\begin{aligned}|y-\varphi_i(\alpha(n))|\ge\frac{1}{2}|\varphi_1(\alpha(n))-\varphi_i(\alpha(n))|&=\frac{1}{2}|(\varphi_1-\varphi_i)(\alpha(n))|\\ &>a\cdot|\alpha(n)^{1/d}|\,,\end{aligned}$$

for a certain positive constant $a>0$.

If there exist some $i\in\{2,\dots,d\}$ such that $k_i/e_i=k_1/e_1$, but $a_{-k_i i}\ne a_{-k_1 1}$, since $k_1\ge 1$ for these i, for n large enough we have

$$\begin{aligned}|y-\varphi_i(\alpha(n))|&>\frac{1}{2}|\varphi_1(\alpha(n))-\varphi_i(\alpha(n))|\\ &>f\cdot|a_{-k_1 1}\alpha(n)^{k_1/e_1}-a_{-k_i i}\alpha(n)^{k_1/e_1}|\\ &=f\cdot|a_{-k_1 1}-a_{-k_i i}|\cdot|\alpha(n)^{k_1/e_1}|\\ &>\ g\cdot|\alpha(n)^{1/e_1}|\\ &\ge g\cdot|\alpha(n)^{1/d}|,\end{aligned}$$

for certain positive constants f and g.

Therefore, for every $i=2,\dots,d$, the inequality

$$|y-\varphi_i(\alpha(n))|\ \ge\ h\cdot|\alpha(n)^{1/d}|, \tag{23}$$

holds for a certain constant $h=\min\{a,g\}$.

From (23) it follows, with $b = h^{d-1}$, that the inequality

$$\begin{aligned}|F(\alpha(n),y)| &= |y-\varphi_1(\alpha(n))|\cdot|y-\varphi_2(\alpha(n))|\cdot\ldots\cdot|y-\varphi_d(\alpha(n))|\\ &> b\cdot|\alpha(n)^{(d-1)/d-\varepsilon_1}|\\ &= b\cdot|\alpha(n)^{1-\frac{1}{d}-\varepsilon_1}|\\ &= b\cdot|\alpha(n)|^{1-\frac{1}{d}-\varepsilon_1}\end{aligned}$$

holds for all pairs $(n,y)\in\mathbb{N}\times\mathbb{Z}$ with n large enough and $y\neq\beta_i(n)$ for every $i=1,\ldots,t$.

Choosing $\varepsilon_1>0$ small enough we obtain, for n large enough

$$b\cdot|\alpha(n)|^{1-\frac{1}{d}-\varepsilon_1} > c\cdot|\alpha(n)|^{1-\frac{1}{d}-\varepsilon}.$$

Therefore the inequality

$$|F(\alpha(n),y)| > c\cdot|\alpha(n)|^{1-\frac{1}{d}-\varepsilon}$$

holds for all the pairs $(n,y)\in\mathbb{N}\times\mathbb{Z}$ with n large enough and $y\neq\beta_i(n)$ for every $i=1,\ldots,t$.

This means that each solution of (3) has $y=\beta_i(n)$, for a certain $i=1,\ldots,s$, with $s\geq t$.

As in the proof of Theorem 3.1, we can obtain that the natural numbers n such that (n,y) is a solution of the inequality, except finitely many, make up a finite union of arithmetical progressions and that the roots of the power sums $\beta_1(n),\ldots,\beta_s(n)$ are positive integers lying in the multiplicative group generated by the real e-th roots, with $1\leq e\leq d$, of the roots of the power sum $\alpha(n)$.

Since the set $\{\beta_1(n),\ldots,\beta_t(n)\}$ contains at most d non constant power sums, and since we have d choices for the set M_i, the set $\{\beta_1(n),\ldots,\beta_s(n)\}$ contains at most d^2 non constant power sums. □

Remark 5.1. From Corollary 3.2 we can derive that if the inequality (3) has infinitely many solutions, then there exists at least one power sum $\beta(n)\in\Sigma_{\mathbb{Z}}^{+}$ not constant such that $(n,\beta(n))$ is a solution. Since $F(\alpha(n),\beta(n))$ is a power sum, to have infinitely many solutions to (3) the absolute value of the largest root of $F(\alpha(n),\beta(n))$ must be smaller than $|c_1|^{1-\frac{1}{d}-\varepsilon}$, where c_1 is the largest root of α. This means that to have infinitely many solutions the coefficients of the roots of the power sum $F(\alpha(n),\beta(n))$ with absolute value larger than $|c_1|^{1-\frac{1}{d}-\varepsilon}$ must vanish. This condition is easily verifiable in concrete cases with algebrical methods, so it is easy to decide wheather the inequality (3), with a particular power sum $\alpha(n)$, a particular polynomial F and a particular value of ε, has infinitely many solutions or not.

Remark 5.2. If the polynomial F is not regular, we can get a weaker result than that of Corollary 3.2. Using the same notations of Corollary 3.2, let

$$\overline{d} = \max_{i=1,\ldots,d} \{ \, |\{\varphi_j : k_j/e_j = k_i/e_i \quad \text{and} \quad a_{k_j j} = a_{k_i i}\}| \, \}. \tag{24}$$

If F is not regular we have $2 \leq \overline{d} \leq d$. Without losing generality, let $\varphi_1, \ldots, \varphi_{\overline{d}}$ be the $\overline{d}$ Puiseux series such that $k_1/e_1 = \ldots = k_{\overline{d}}/e_{\overline{d}}$o and $a_{k_1 1} = \ldots = a_{k_{\overline{d}}\overline{d}}$.

As in the proof of Corollary 3.2, we obtain that

$$\begin{aligned} |F(\alpha(n), y)| &= |y - \varphi_1(\alpha(n))| \cdot \ldots \cdot |y - \varphi_{\overline{d}}(\alpha(n))| \\ &\quad \cdot |y - \varphi_{\overline{d}+1}(\alpha(n))| \cdot \ldots \cdot |y - \varphi_d(\alpha(n))| \\ &> c \cdot |\alpha(n)|^{-\varepsilon_1 \overline{d}} \cdot \left(|\alpha(n)|^{1/d}\right)^{d-\overline{d}} \\ &= c \cdot |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon} \end{aligned}$$

holds for all the pairs (n, y) such that $y \neq \beta_i(n)$ for every $i = 1, \ldots, s$, where $\{\beta_1(n), \ldots, \beta_s(n)\}$ is a finite set of power sums with positive integral roots and rational coefficients.

So for every $c > 0$ and for every $\varepsilon > 0$ fixed, every solution $(n, y) \in \mathbb{N} \times \mathbb{Z}$ of the inequality

$$|F(\alpha(n), y)| < c \cdot |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon} \tag{25}$$

has $y = \beta_i(n)$, for a certain $i \in \{1, \ldots, s\}$.

Let us notice that if $\overline{d} \neq d$, there exist $\varepsilon > 0$ such that $1 - \frac{\overline{d}}{d} - \varepsilon > 0$.

Remark 5.3 If, under the notations of Corollary 3.2 and Remark 5.2, we have $\overline{d} = d$, with a proper substitution we can reduce the polynomial $F(x, y)$ to the cases considered above. Indeed, writing the series of the Puiseux expansion of $F(x, y)$ as

$$\varphi_j(x) = a_{-k}\, x^{k/e_j} + \ldots + a_{-g}\, x^{g/e_j} + \sum_{i=-g+1}^{+\infty} a_{ij}\, x^{-i/e_j},$$

with $j = 1, \ldots, d$, where a_{-g} is the last common term in every $\varphi_j(x)$, we have

$$F(x, y) = \prod_{j=1}^{d} \Bigg(y - \sum_{i=-k}^{-g} a_i\, x^{-i/e_j} - \sum_{i=-g+1}^{+\infty} a_{ij}\, x^{-i/e_j} \Bigg).$$

Applying the substitution

$$y - \sum_{i=-k}^{-g} a_i\, x^{-i/e_j} \longmapsto z,$$

we obtain a new polynomial $G(x, z)$ that, for the choice of the substitution, can either be regular, and so we can apply Corollary 3.2, or satisfy the hypothesis of Remark 5.2.

Proof of Corollary 3.3. Let $\overline{d}$ be defined as in (24). We can have that either the inequality

$$|F(\alpha(n), y)| < |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon}, \tag{26}$$

with $\varepsilon = \frac{1}{2d}$, has finitely many solutions $(n, y) \in \mathbb{N} \times \mathbb{Z}$ or infinitely many.

If (26) has only finitely many solutions, let us observe that, since $\alpha(n)$ is not constant, for n large enough we have

$$2\ |f(n)|\ <\ |\alpha(n)|^{1-\frac{\overline{d}}{d}-\varepsilon},$$

and so also the inequality $|F(\alpha(n), y)| < 2\ |f(n)|$ has finitely many solutions.

The solutions of $F(\alpha(n), y) = f(n)$ are contained in the set of solutions of $|F(\alpha(n), y)| < 2\ |f(n)|$, and so they are only finitely many.

If (26) has infinitely many solutions, from Theorem 3.1 (if $F(x, y)$ is regular), Remark 5.2 (if $\overline{d} < d$) and Remark 5.3 (if $\overline{d} = d$) we know that they all have $y = \beta_i(n)$, for $i = 1, \ldots, s$, where $\{\beta_1, \ldots, \beta_s\}$ is a set of power sums with rational coefficients and positive integral roots.

For every $i = 1, \ldots, s$, $F(\alpha(n), \beta_i(n))$ is a power sum that may be constant.

If for a certain i $F(\alpha(n), \beta_i(n))$ is constant, we have

$$\frac{F(\alpha(n), \beta_i(n))}{f(n)} \xrightarrow{n\to\infty} 0.$$

If for a certain i $F(\alpha(n), \beta_i(n))$ is not constant, we have

$$\left|\frac{F(\alpha(n), \beta_i(n))}{f(n)}\right| \xrightarrow{n\to\infty} +\infty.$$

In both cases $F(\alpha(n), \beta_i(n))$ can not assume the values of $f(n)$ for infinitely many n, and so the equation $F(\alpha(n), y) = f(n)$ has only finitely many solutions. □

Remark 5.4. In Corollary 3.3 the assumption that $|\alpha(n)|$ is not constant is necessary. Consider e.g. the case $\alpha(n) = 1$, $F(x, y) = y^2 + x$, $f(n) = n^2 + 1$, that has as solutions the couples $(n, \pm n)$, $n \in \mathbb{N}$. In all the other statements of the present paper this assumption is not required.

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