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On multiple analogues of Ramanujan's formulas for certain Dirichlet series

par Hirofumi TSUMURA

Résumé. Dans cet article, nous prouvons des analogues multiples des célèbres formules de Ramanujan pour certaines séries de Dirichlet, qui ont été présentées dans ses cahiers bien connus. De plus, nous obtenons des versions multiples de formules semblables à celles de Ramanujan qui ont été données par Berndt et d'autres auteurs.

Abstract. In this paper, we prove multiple analogues of famous Ramanujan's formulas for certain Dirichlet series which were introduced in his well-known notebooks. Furthermore, we prove some multiple versions of analogous formulas of Ramanujan which were given by Berndt and so on.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{C}$ the field of complex numbers.

Ramanujan obtained the following fascinating formula:

$$\begin{aligned}\alpha^{-N}&\left\{\frac{1}{2}\zeta(2N+1)+\sum_{k=1}^{\infty}\frac{1}{(e^{2k\alpha}-1)\,k^{2N+1}}\right\}\\ &=(-\beta)^{-N}\left\{\frac{1}{2}\zeta(2N+1)+\sum_{k=1}^{\infty}\frac{1}{(e^{2k\beta}-1)\,k^{2N+1}}\right\}\\ &\quad-2^{2N}\sum_{k=0}^{N+1}(-1)^k\frac{B_{2k}}{(2k)!}\frac{B_{2N+2-2k}}{(2N+2-2k)!}\alpha^{N+1-k}\beta^k,\end{aligned} \tag{1.1}$$

where N is any non-zero integer, α and β are positive numbers such that $\alpha\beta = \pi^2$, $\zeta(s)$ is the Riemann zeta-function, and $\{B_n\}$ are the Bernoulli

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numbers defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

The formula (1.1) was introduced in Ramanujan's notebooks [5, P. 275], and its proof was given by several authors (for details, see, for example, [1, 3, 4, 9]). Let

$$\mathcal{X}(s):=\sum_{k=1}^{\infty}\frac{1}{(e^{2k\pi}-1)\,k^s}\quad(s\in\mathbb{C}). \tag{1.2}$$

If we put $\alpha=\beta=\pi$ in (1.1) and consider the case $s=N$ for any odd positive N, then the formulas

$$\begin{aligned}\mathcal{X}(3)&=-\frac{1}{2}\zeta(3)+\frac{7}{360}\pi^3;\\ \mathcal{X}(7)&=-\frac{1}{2}\zeta(7)+\frac{19}{113400}\pi^7;\\ \mathcal{X}(11)&=-\frac{1}{2}\zeta(11)+\frac{1453}{851350500}\pi^{11}\end{aligned} \tag{1.3}$$

can be derived. These results have been already studied by Lerch ([10]) and Malurkar ([11]); see also [5].

On the other hand, if we consider the case of any odd negative N, then

$$\begin{aligned}\mathcal{X}(-1)&=\frac{1}{24}-\frac{1}{8\pi};\\ \mathcal{X}(-4m-1)&=\frac{B_{4m+2}}{8m+2}\quad(m\geq 1)\end{aligned} \tag{1.4}$$

can be derived. Furthermore, Ramanujan also obtained

$$\begin{aligned}\mathcal{X}(1)&=-\frac{\pi}{12}-\frac{1}{2}\log\left(\frac{\varpi}{\sqrt{2}\pi}\right);\\ \mathcal{X}(-3)&=\frac{\Gamma(1/4)^8}{5120\pi^6}-\frac{1}{240}=\frac{1}{80}\left(\frac{\varpi}{\pi}\right)^4-\frac{1}{240},\end{aligned} \tag{1.5}$$

where

$$\varpi=2\int_0^1\frac{dx}{\sqrt{1-x^4}}.$$

Moreover, we consider

$$\mathfrak{X}_1(s)=\sum_{k=1}^{\infty}\frac{\coth(k\pi)}{k^s}\quad(s\in\mathbb{C};\ \Re s>1), \tag{1.6}$$

where $\coth(x) = (e^{2x}+1)\,/\,(e^{2x}-1)$. Then we see that for any $s \in \mathbb{C}$ with $\Re s > 1$,

$$\mathfrak{X}_1(s) = \sum_{k=1}^{\infty} \frac{e^{2k\pi}-1+2}{(e^{2k\pi}-1)\,k^s} = 2\mathcal{X}(s) + \zeta(s). \tag{1.7}$$

Hence $\mathfrak{X}_1(s)$ can be meromorphically continued to $\mathbb{C}$. By combining (1.1) and (1.7), we have

$$\mathfrak{X}_1(4k+3) = \frac{(2\pi)^{4k+3}}{2} \sum_{j=0}^{2k+2} (-1)^{j+1} \frac{B_{2j}}{(2j)!} \frac{B_{4k+4-2j}}{(4k+4-2j)!} \tag{1.8}$$

for $k \in \mathbb{N} \cup \{0\}$ (see also Ramanujan's notebooks [5, (25.3) P. 293]).

Needless to say, the fascination of these formulas is common with that of Euler's well-known formula for $\zeta(2k)$. For these 15 years, as a multiple analogue of $\zeta(s)$, the multiple zeta value

$$\sum_{k_1,\ldots,k_r=1}^{\infty} \frac{1}{k_1^{p_1}(k_1+k_2)^{p_2}\cdots(k_1+\cdots+k_r)^{p_r}} \qquad (p_1,\ldots,p_r \in \mathbb{N}; p_r \geq 2)$$

(see, for example, [8, 13]) has been investigated actively in various branches of mathematics and physics, and many fascinating formulas among multiple zeta values were obtained. Therefore it seems to be interesting to consider certain multiple analogues of $\mathfrak{X}_1$. Indeed, in [12], Matsumoto and the author gave some functional relation formulas, for example,

$$\sum_{\substack{k,l \in \mathbb{Z}\setminus\{0\} \\ k+l>0}} \frac{(-1)^{k+l+1}}{\sinh(k\pi)k^{2p}(k+l)^s} = -\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sinh(k\pi)k^{s+2p}} \tag{1.9}$$

for $s \in \mathbb{C}$, $p \in \mathbb{Z}$.

In this paper, as r-ple analogues of $\mathfrak{X}_1$, we consider

$$\begin{aligned} &\mathfrak{X}_r(2p_1,\ldots,2p_{r-1},s) \\ &:= \sum_{\mathbf{k}\in\Phi_r} \frac{\coth(k\pi)}{\prod_{d=1}^{r-1}\left(\sum_{j=1}^{d} k_j\right)^{2p_d} \cdot \left(\sum_{j=1}^{r} k_j\right)^{s}} \end{aligned} \tag{1.10}$$

for $r \in \mathbb{N}$ with $r \geq 2$ and $p_1, p_2, \ldots, p_{r-1} \in \mathbb{N}$, where

(1.11)

$$\Phi_r = \left\{ \mathbf{k} = (k_j) \in (\mathbb{Z}\setminus\{0\})^r \;\middle|\; \sum_{j=1}^{d} k_j \neq 0 \ (1 \leq d \leq r-1),\ \sum_{j=1}^{r} k_j > 0 \right\}.$$

Using the method similar to our previous one, we prove some functional relations for these series. For example, we have

$$\mathfrak{X}_2(2p,s) = -\mathfrak{X}_1(s+2p) \quad (s \in \mathbb{C};\, p \in \mathbb{N}), \tag{1.12}$$

which is an analogue of (1.9). Putting $p=1$ and $s=5$ in (1.12), we have

$$\mathfrak{X}_2(2,5) = -\mathfrak{X}_1(7) = -\frac{19}{56700}\pi^7.$$

Also, putting $s=-3$ and using (1.4), we have

$$\mathfrak{X}_2(2,-3) = -\mathfrak{X}_1(-1) = -2\mathcal{X}(-1) - \zeta(-1) = \frac{1}{4\pi}$$

in the non-convergent area. These can be regarded as certain multiple analogues of Ramanujan's formulas for the values not only at positive integers but also at negative integers.

2. Key lemma

In this section, we prove the following lemma.

Lemma 2.1. *For $r \in \mathbb{N}$ with $r \geq 2$, $p_1, p_2, \ldots, p_r \in \mathbb{N}$ and $\theta \in \mathbb{R}$,*

$$\sum_{\mathbf{k}\in\Phi_r} \frac{\coth(k_1\pi)\sin\left(\left(\sum_{j=1}^r k_j\right)\theta\right)}{\prod_{d=1}^r \left(\sum_{j=1}^d k_j\right)^{2p_d}} + (-1)^r \sum_{k=1}^{\infty} \frac{\coth(k\pi)\sin(k\theta)}{k^{2\sum_{d=1}^r p_d}} = 0. \tag{2.1}$$

Proof. We prove the assertion by induction on $r \geq 2$. Let

$$\Lambda_r = \left\{ \mathbf{k} = (k_j) \in (\mathbb{Z}\setminus\{0\})^{r-1}\times\mathbb{Z} \,\middle|\, \sum_{j=1}^d k_j \neq 0 \ (1 \leq d \leq r) \right\}. \tag{2.2}$$

In the case $r=2$, we see that

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\Lambda_2} \frac{\coth(k_1\pi)\sin((k_1+k_2)\theta)}{k_1^{2p_1}(k_1+k_2)^{2p_2}} \\
&= \sum_{k_1\in\mathbb{Z}\setminus\{0\}} \sum_{\substack{k_2\in\mathbb{Z}\\ k_1+k_2\neq 0}} \frac{\coth(k_1\pi)\sin((k_1+k_2)\theta)}{k_1^{2p_1}(k_1+k_2)^{2p_2}} \\
&= \sum_{k_1\in\mathbb{Z}\setminus\{0\}} \frac{\coth(k_1\pi)}{k_1^{2p_1}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\sin(n\theta)}{n^{2p_2}} = 0
\end{aligned}$$

by putting $n = k_1 + k_2$ and changing the index n into $-n$. The left-hand side is equal to

$$\sum_{k_1\in\mathbb{Z}\setminus\{0\}} \sum_{\substack{k_2\in\mathbb{Z}\setminus\{0\}\\ k_1+k_2\neq 0}} \frac{\coth(k_1\pi)\sin((k_1+k_2)\theta)}{k_1^{2p_1}(k_1+k_2)^{2p_2}} + \sum_{k_1\in\mathbb{Z}\setminus\{0\}} \frac{\coth(k_1\pi)\sin(k_1\theta)}{k_1^{2p_1+2p_2}}$$
$$= 2\sum_{(k_1,k_2)\in\Phi_2} \frac{\coth(k_1\pi)\sin((k_1+k_2)\theta)}{k_1^{2p_1}(k_1+k_2)^{2p_2}} + 2\sum_{k_1=1}^{\infty} \frac{\coth(k_1\pi)\sin(k_1\theta)}{k_1^{2p_1+2p_2}},$$

by changing the indices (k_1, k_2) into $(-k_1, -k_2)$. Thus we have the assertion in the case $r = 2$.

Now we assume the case of $r-1$ holds and consider the case of r.

$$\sum_{(k_j)\in\Lambda_r} \frac{\coth(k_1\pi)\sin\left(\left(\sum_{j=1}^{r} k_j\right)\theta\right)}{\prod_{d=1}^{r}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}}$$
$$= \sum_{\substack{(k_j)\in\Lambda_{r-1}\\ k_{r-1}\neq 0}} \frac{\coth(k_1\pi)}{\prod_{d=1}^{r-1}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}} \sum_{n\neq 0} \frac{\sin(n\theta)}{n^{2p_r}} = 0$$

by putting $n = k_1 + \cdots + k_r$ and changing the index n into $-n$. The left-hand side is equal to

$$2\sum_{(k_j)\in\Phi_r} \frac{\coth(k_1\pi)\sin\left(\left(\sum_{j=1}^{r} k_j\right)\theta\right)}{\prod_{d=1}^{r}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}}$$
$$+ 2\sum_{(k_j)\in\Phi_{r-1}} \frac{\coth(k_1\pi)\sin\left(\left(\sum_{j=1}^{r-1} k_j\right)\theta\right)}{\prod_{d=1}^{r-2}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}\cdot\left(\sum_{j=1}^{r-1} k_j\right)^{2p_{r-1}+2p_r}}$$

by sparating $k_r \neq 0$ or $= 0$ and by changing the indices $(k_1, \ldots, k_r)$ into $(-k_1, \ldots, -k_r)$. By the assumption of induction, the second term is equal to

$$-2(-1)^{r-1}\sum_{k=1}^{\infty} \frac{\coth(k\pi)\sin(k\theta)}{k^{2\sum_{d=1}^{r} p_d}}.$$

Thus we have the assertion in the case of r. By induction, we complete the proof of this lemma. □

3. Main theorem and its proof

In this section, we prove the main theorem in this paper as follows.

Theorem 3.1. *For $r \in \mathbb{N}$ with $r \geq 2$ and $p_1, p_2, \ldots, p_{r-1} \in \mathbb{N}$,*

$$\mathfrak{X}_r(2p_1, \ldots, 2p_{r-1}, s) = (-1)^{r-1}\mathfrak{X}_1\left(s + 2\sum_{j=1}^{r-1} p_j\right) \tag{3.1}$$

holds for $\Re s > 1$. Furthermore the left-hand side of (3.1) *can be continued meromorphically to $\mathbb{C}$ by this equation.*

First we quote the following fact which is one of the main result in our previous paper [12].

Let $Z(s)$ be the Dirichlet series defined by

$$Z(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \tag{3.2}$$

where $\{a_n\} \subset \mathbb{C}$. Let $\Re s = \rho$ ($\rho \in \mathbb{R}$) be the abscissa of convergence of $Z(s)$. This means that if $\Re s > \rho$ then $Z(s)$ is convergent and if $\Re s < \rho$ then $Z(s)$ is not convergent. We further assume that $0 \leq \rho < 1$.

Theorem 3.2 ([12] Theorem 3.1)**.** *Assume that*

$$\sum_{m=1}^{\infty} a_m \sin(mt) = 0 \tag{3.3}$$

is boundedly convergent for $t > 0$ and that, for $\rho < s < 1$,

$$\lim_{\lambda\to\infty} \sum_{m=1}^{\infty} a_m \int_{\lambda}^{\infty} t^{s-1}\sin(mt)dt = 0. \tag{3.4}$$

Then $Z(s)$ can be continued meromorphically to $\mathbb{C}$, and actually $Z(s) = 0$ for all $s \in \mathbb{C}$.

Now we aim to apply this theorem to the situation mentioned above. For this aim, we further quote the following result.

Lemma 3.3 ([12] Lemma 7.6)**.** *For $r \in \mathbb{N}$ and $p_1, \ldots, p_r \in \mathbb{N}$,*

$$\sum_{\mathbf{k}\in\Omega_r} \frac{1}{\prod_{d=1}^{r}\left(\sum_{j=1}^{d} k_j\right)^{2p_d}} \leq \prod_{d=1}^{r}\{2\zeta(2p_d)\} < \infty,$$

where

$$\Omega_r := \left\{\mathbf{k} = (k_j) \in (\mathbb{Z}\setminus\{0\})^r \;\middle|\; \sum_{j=1}^{d} k_j \neq 0 \ (1 \leq d \leq r)\right\}.$$

Proof of Theorem 3.1. By Lemma 3.3, we see that the left-hand side of (2.1) satisfies the conditions (3.3) and (3.4) because of the absolute convergency. Hence, by Theorem 3.2, we see that

$$\mathfrak{X}_r(2p_1,\ldots,2p_{r-1},2p_r+s)+(-1)^r\mathfrak{X}_1\left(s+2\sum_{j=1}^{r}p_j\right)=0 \tag{3.5}$$

holds for any $s\in\mathbb{C}$, where the left-hand side of (3.5) is continued meromorphically to $\mathbb{C}$. Since $\mathfrak{X}_1(s)$ is continued meromorphically to $\mathbb{C}$, so is $\mathfrak{X}_r(2p_1,\ldots,2p_{r-1},2p_r+s)$. Hence, by placing $2p_r+s$ with s, we have the assertion of Theorem 3.1. □

Example 1. By Theorem 3.1, we have, for example,

$$\mathfrak{X}_2(2,s)=-\mathfrak{X}_1(s+2); \tag{3.6}$$

$$\mathfrak{X}_3(2,6,s)=\mathfrak{X}_1(s+8); \tag{3.7}$$

$$\mathfrak{X}_4(4,4,2,s)=-\mathfrak{X}_1(s+10). \tag{3.8}$$

Putting $s=3$ in (3.7) and $s=5$ in (3.8), we respectively obtain

$$\mathfrak{X}_3(2,6,3)=\mathfrak{X}_1(11)=\frac{1453}{425675250}\pi^{11};$$
$$\mathfrak{X}_4(4,4,2,5)=-\mathfrak{X}_1(15)=-\frac{13687}{390769879500}\pi^{15}.$$

Also, by putting $s=-3$ in (3.6) and by using (1.5) and (1.7), we have

$$\mathfrak{X}_2(2,-5)=-\mathfrak{X}_1(-3)=-2\mathcal{X}(-3)-\zeta(-3)$$
$$=-\frac{\Gamma(1/4)^8}{2560\pi^6}=-\frac{1}{40}\left(\frac{\varpi}{\pi}\right)^4.$$

We can regard these results as multiple analogues of Ramanujan's formulas for the values not only at positive integers but also at negative integers.

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