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Class invariants and cyclotomic unit groups from special values of modular units

par Amanda FOLSOM

Résumé. Dans cet article, nous obtenons des invariants de classe et des groupes d'unités cyclotomiques en considérant des spécialisations d'unités modulaires. Nous construisons ces unités modulaires à partir de solutions d'équations fonctionnelles de q-récurrence données par Selberg dans son travail généralisant les identités de Rogers-Ramanujan. Commme corollaire, nous donnons une nouvelle preuve d'un résultat de Zagier et Gupta, originellement considéré par Gauss, à propos des périodes de Gauss. Ces résultats proviennent pour partie de la thèse de l'auteur en 2006 [6] dans laquelle la structure de ces groupes d'unités modulaires et de leur groupe de classes de diviseurs cuspidaux associé est donnée en termes de produits de fonctions L et comparée à la formule classique du nombre de classes relatives pour les corps cyclotomiques [6, 7].

Abstract. In this article we obtain class invariants and cyclotomic unit groups by considering specializations of modular units. We construct these modular units from functional solutions to higher order q-recurrence equations given by Selberg in his work generalizing the Rogers-Ramanujan identities. As a corollary, we provide a new proof of a result of Zagier and Gupta, originally considered by Gauss, regarding the Gauss periods. These results comprise part of the author's 2006 Ph.D. thesis [6] in which the structure of these modular unit groups and their associated cuspidal divisor class groups are also characterized, and a cuspidal divisor class number formula is given in terms of products of L-functions and compared to the classical relative class number formula within the cyclotomic fields [6, 7].

1. Introduction and statement of results.

Let $\mathcal{F}_\ell$, $\ell \in \mathbb{N}$, be the modular function field with respect to the principal congruence subgroup $\Gamma(\ell) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv 1 \bmod \ell\}$ defined over

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$\mathbb{Q}(\zeta_\ell)$, where $\zeta_\ell = e^{2\pi i/\ell}$. The modular functions that comprise the invertible elements in the integral closure of the ring $\mathbb{Q}[j] \subset \mathcal{F}_\ell$ form a group, the modular unit group of level ℓ, where $j = j(\tau) = q^{-1} + 744 + 196884q + \cdots$ is the classical modular invariant.

In this paper we define modular units $r_{\ell,j}(\tau), 1 \leq j \leq (\ell-3)/2$ for $\ell = 2k+1 \geq 5$, and the groups they generate

$$U_\ell^C := \langle \{ r_{\ell,j}(\tau) \mid 1 \leq j \leq (\ell-3)/2 \} \rangle \subset U_\ell \tag{1.1}$$

using functional solutions to higher order q-difference equations found in the work of Selberg [26], where in (1.1) and what follows, $\langle x \rangle$ denotes the group generated by x. We will explicitly define the groups U_ℓ^C and their generators $r_{\ell,j}$, $1 \leq j \leq (\ell-3)/2$ in §3. To obtain class invariants we consider special values of the modular units $r_{\ell,j}$, $1 \leq j \leq (\ell-3)/2$, at points of complex multiplication, and show that only a single value of one such modular unit is needed to generate class fields of prime moduli, to which we compare the classical theory of complex multiplication. We then generalize these results to class fields of more arbitrary moduli, motivated by the work of Ramachandra, combining the theory of Kubert and Lang with that of Shimura.

With respect to the cyclotomic theory, we portray these modular units as modular function field analogues to cyclotomic units, and show that when specialized to 0 these modular units yield cyclotomic units that may be used to generate cyclotomic unit groups. Dually, within the modular function fields we show how these modular units may be used as generators. As a corollary, we provide a new proof of a result of Zagier and Gupta, originally considered by Gauss. Let

$$S_\ell^C := \{ r_{\ell,j}(\tau) \mid 1 \leq j \leq (\ell-3)/2 \}. \tag{1.2}$$

Our main results are the following.

Theorem 1.1.

i. For an odd integer $\ell = 2k+1 \geq 5$, the functions $r_{\ell,j}(\tau)$, $1 \leq j \leq (\ell-3)/2$ defined in (3.15), (3.16) and (3.20) are modular units of level ℓ.

ii. Further, for $\ell > 5$, the modular units $r_{\ell,1}$ and $r_{\ell,2}$ generate the field of modular functions $\mathcal{F}_\ell$. That is,

$$\mathcal{F}_\ell = \mathbb{Q}(\zeta_\ell, r_{\ell,1}, r_{\ell,2}).$$

In the case $\ell = 5$,

$$\mathcal{F}_5 = \mathbb{Q}(\zeta_5, r_{5,1}).$$

Theorem 1.2. *For $\ell = 2k+1 \geq 5$, the set of modular units S_ℓ^C of level ℓ are such that their limiting values in the cusp 0 are cyclotomic units in $\mathbb{Q}(\zeta_\ell)^+$. That is,*

$$\lim_{\tau\to 0} S_\ell^C := \left\{ \lim_{\tau\to 0} r_{\ell,j}(\tau) \mid 1 \leq j \leq k-1 \right\} \subseteq \mathrm{E}^{\mathrm{C}}_{\mathbb{Q}(\zeta_\ell)^+}. \tag{1.3}$$

Further, if ℓ is prime, then we have equality in (1.3), that is

$$\mathrm{E}^{\mathrm{C}}_{\mathbb{Q}(\zeta_\ell)^+} = \left\langle \lim_{\tau\to 0} S_\ell^C \right\rangle.$$

Theorem 1.3. *Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K = \mathbb{Z}[\tau], \tau \in \mathcal{H}$ an algebraic integer. Then for any prime $\ell \geq 5$ and any integer m such that $1 \leq m \leq (\ell-3)/2$, the ray class field K_ℓ of modulus ℓ over the Hilbert class field K_H is given by*

$$K_\ell = K_H(r_{\ell,m}(\tau)).$$

In particular, if $j(\tau) \in K(r_{\ell,m}(\tau))$, then

$$K_\ell = K(r_{\ell,m}(\tau)).$$

We generalize the results of Theorem 1.3 to class fields of more arbitrary moduli $\mathfrak{f}$, with generalized invariants $\Theta_\mathfrak{f}$ that reduce in a special case to powers of the (specializations of the) modular units $r_{\ell,j}(\tau)$. In what follows, $\mathbf{N}(\tau)$ and $\mathbf{T}(\tau)$ respectively refer to the norm and trace of $\tau \in \mathcal{H}$, and the normalized Klein form φ is given by

$$\varphi\left(z \,\middle|\, \begin{matrix}\omega_1\\ \omega_2\end{matrix}\right) = 2\pi i e^{-zz^*/2}\sigma(z|L)\eta(\omega_1/\omega_2)^2\omega_2^{-1},$$

where $z = z_1\omega_1 + z_2\omega_2$ is given by the real coordinates z_1 and z_2, $z^* = z_1\eta_1 + z_2\eta_2$, and η_1, η_2 are the quasi-periods of the elliptic Weierstrass ζ-function of the lattice L belonging to ω_1, ω_2. For the statement of Theorems 1.4 and 1.5, we let K be an imaginary quadratic field of discriminant d with ring of integers $\mathcal{O}_K = [\tau, 1]$, and impose the following hypotheses **H0** – **H5** (as in [3]):

H0: $\tau \in \mathcal{H}$ is chosen to satisfy $\mathbf{T}(\tau) \equiv 0 \mod 3$,
$$\mathbf{T}(\tau) \equiv \begin{cases} 0 \mod 4 & \text{if } 2|d, \\ 1 \mod 4 & \text{if } 2 \nmid d. \end{cases}$$
(Note for a given K, such a choice of τ is always possible.)

H1: $\mathfrak{f} \subseteq \mathcal{O}_K$ is an ideal of the form $s_2[\tau + m_\tau, s_1]$ where s_1, s_2, m_τ are integers such that $\gcd(s_1, s_2) = 1, s_1|\mathbf{N}(\tau),\ s_1|m_\tau + \mathbf{T}(\tau), s_1s_2 \nmid 6$, and s_1, s_2 prime integers whose product $s_1s_2 = \ell \in \mathbb{N}$.

H2: $\xi \in K^*$ is of the form $\xi = \frac{r_1}{s_1}\tau + \frac{r_2}{s_2} \in K^*$, where $\gcd(r_1, s_1) = \gcd(r_2, s_2) = 1$.

H3: $\mathfrak{f}$ decomposes as $\mathfrak{f} = f_1\mathfrak{f}_2$ where $f_1 \in \mathbb{N}$ and $\mathfrak{f}_2$ is a primitive ideal of norm f_2, and f_2^* and f_2^{**} denote the split part and non-split part, respectively, of f_2.

H4: For $1 \le i \le m+1$, $m \in \mathbb{Z}^+$, $n_i \in \mathbb{Z}$, the ideals $\mathfrak{b}_i \subseteq \mathcal{O}_K$ are defined by $\overline{\mathfrak{b}_i} = [\tau, b_i]$, $1 \le i \le m$, and for $i \ne m+1$, $\overline{\mathfrak{b}_i\mathfrak{b}_{m+1}} = [\tau, b_i b_{m+1}]$, where for all $i, 1 \le i \le m+1$, $b_i \in \mathbb{Z}$, $\gcd(b_i, 6\ell) = 1$.

H5: Let $\tilde{\tau} \in \mathcal{O}_K$, with $\mathrm{Im}(\tilde{\tau}) > 0$. Then $\gcd(\mathbf{T}(\tilde{\tau}), f_2^{**}) = 1$ so there is some a satisfying

$$\mu a \mathbf{T}(f_1\tilde{\tau}) - \mathbf{N}(\mathfrak{f}\xi)\sum_{i=1}^{m} n_i b_i \mathbf{N}(\lambda_i) \equiv 0 \bmod 2\ell,$$

where $\mu = 1$ if f_2^{**} is even, and $\mu = 2$ if f_2^{**} is odd. With this notation we set $\zeta := \exp(2\pi i\mu a/2f)$.

Theorem 1.4. *Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K = [\tau, 1]$, $\tau \in \mathcal{H}$ satisfying* H0*, and let $\mathfrak{f}$, ξ $\mathfrak{b}_i$ satisfy* H1, H2, *and* H4. *If $\prod b_i \mid \mathbf{N}(\tau)$, then the product*

$$\Theta_{\mathfrak{f}} := \prod_{i=1}^{m} \varphi(\xi | \mathfrak{f}\mathfrak{b}_i^{-1})^{12\ell n_i} \tag{1.4}$$

generates the ray class field of modulus $\mathfrak{f}$ over the Hilbert class field K_H. That is,

$$K_H(\Theta_{\mathfrak{f}}) = K_{\mathfrak{f}}. \tag{1.5}$$

In particular, if $j(\tau) \in K(\Theta_{\mathfrak{f}})$, then

$$K(\Theta_{\mathfrak{f}}) = K_{\mathfrak{f}}.$$

Further, if we do not require $\prod b_i \mid \mathbf{N}(\tau)$, we have the product $\Theta_{\mathfrak{f}} \in K_{\mathfrak{f}}$.

A result in [24] asserts that under various conditions on the decomposition of a modulus $\mathfrak{f} = \mathbb{Z}\tau + \mathbb{Z}$ into prime ideals, and under certain hypotheses[1] placed on an ideal class $\mathfrak{b}$ in the ray class group, where $\mathfrak{f}\mathfrak{b}^{-1} = \mathbb{Z}\tau b^{-1} + \mathbb{Z}$, $b \in \mathbb{Z}$, the simple quotient of the normalized Klein forms $\big((\varphi(1|\mathfrak{f}\mathfrak{b}^{-1})/\varphi(1|\mathfrak{f})\big)^{12\ell}$ and the specialization $\varphi(1|\mathfrak{f})^{12\ell}$ may be used to generate $K_{\mathfrak{f}}$ over K. The authors conjecture that this result holds for arbitrary conductor $\mathfrak{f}$ and every ideal $\mathfrak{b}$ prime to $\mathfrak{f}$ whose ideal class in the ray class group is non trivial.

In [3], the authors form a more general product of normalized Klein forms and provide hypotheses under which a product of *lower powers* of

[1] We refer the reader to [24] for the complete theorem and hypotheses.

such forms multiplied by an explicitly given root of unity ζ lies in the ray class field:

$$\zeta \prod_{i=1}^{m} \varphi\Big(\xi\lambda_i \;\Big|\; \begin{smallmatrix}\tau b_i^{-1}\\ 1\end{smallmatrix}\Big)^{n_i} \in K_{\mathfrak{f}}. \tag{1.6}$$

Note that Θ is essentially a 12^{th} root of $\Theta_{\mathfrak{f}}$. The result (1.6) (with corrected hypotheses as pointed out in [3]) is a generalization of the aforementioned result in [24] regarding $\big(\big(\varphi(1|\mathfrak{f}\mathfrak{b}^{-1})/\varphi(1|\mathfrak{f})\big)^{12\ell}$ and $\varphi(1|\mathfrak{f})^{12\ell}$.

The authors conjecture ([3], p. 341) that with an additional set of hypotheses, the product (1.6) in fact generates $K_{\mathfrak{f}}$ over K. We prove in part these conjectures of [3] and [24]. In Theorem 1.5 that follows, we assume the hypotheses of Theorem 1.4, as well as those of ([3], Theorem 1), which restate for completeness.

Theorem 1.5. *Let K be an imaginary quadratic field of discriminant d with ring of integers $\mathcal{O}_K = [\tau, 1]$ satisfying* H0, *and elements $\mathfrak{f}$, ξ, $\mathfrak{b}_i$, ζ satisfying* H1-H5 *above. Further, let $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathcal{O}_K \setminus \mathfrak{f}$ be elements satisfying*

(1) $\gcd((2), (\lambda_i))$ *are equal for all i with* $2 \nmid n_i$
(2) $n_1 b_1 \mathbf{N}(\lambda_1) + \ldots n_m b_m \mathbf{N}(\lambda_m) \equiv 0 \bmod 2\ell/f_2^{**} \gcd(2, f_2^{**})$.

In addition, we require n_i, b_i as in H4 to satisfy

(3) $\gcd(\sum_{i=1}^{m} b_i n_i, \ell) = 1$
(4) $\prod b_i \mid \mathbf{N}(\tau)$
(5) $n_1 + \cdots + n_m \equiv 0 \bmod 2$
(6) $n_1 b_1 + \cdots + n_m b_m \equiv 0 \bmod 4$ *if* $2|d$ *and* $2 \nmid \mathfrak{f}$
(7) $n_1 b_1 + \ldots n_m b_m \equiv 0 \bmod 3$ *if* $3|d$ *and* $3 \nmid \mathfrak{f}$.

Under these conditions, the product

$$\Theta = \zeta \prod_{i=1}^{m} \varphi(\xi|\mathfrak{f}\mathfrak{b}_i^{-1})^{n_i}$$

generates the ray class field $K_{\mathfrak{f}}$ modulus $\mathfrak{f}$ over the Hilbert class field K_H, that is,

$$K_{\mathfrak{f}} = K_H(\Theta).$$

In particular, if $j(\tau) \in K(\Theta)$ then

$$K_{\mathfrak{f}} = K(\Theta).$$

Finally, to further develop the analogy between the modular units defining the groups U_ℓ^C and the cyclotomic units, as a corollary of the following Theorem 1.6, we provide a new proof of a problem considered by Zagier and Gupta, and previously by Gauss, regarding the Gauss periods.

Theorem 1.6. *For an odd integer* $\ell = 2k+1 \geq 3$, *the modular unit* $(-1)^{1-k} r_{\ell,1}(\tau)$ *as defined in* (3.15) *satisfies the functional equation*

$$M_k(X) = X^k + C_{k-1}X^{k-1} + C_{k-2}X^{k-2} + \cdots + C_1X + C_0$$

where

$$C_{k-j} = (-1)^{\lfloor \frac{j+1}{2} \rfloor} s_{k,j-1}(1) q^{j(k-1)(k+1-\ell)/2\ell} \frac{(q^2;q)_{j-1}}{(1-q)^{j-1}} \frac{C_{k,k}(q^j)}{C_{k,k}(q)}, \tag{1.7}$$

$0 \leq j \leq k$,

$$C_{k,k}(z) = \sum_{n=0}^{\infty} (-1)^n z^{kn} q^{(2k+1)\frac{n^2+n}{2} - kn} (1 - z^k q^{(2n+1)k}) \frac{(zq;q)_n}{(q;q)_n}. \tag{1.8}$$

Further, the limiting value of this functional equation in the cusp 0 *gives an integral polynomial*

$$m_k(x) = \lim_{\tau \to 0} M_k(X) = x^k + c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \cdots c_1 x + c_0$$

with coefficients given by

$$c_{k-j} = (-1)^{\lceil j/2 \rceil} \binom{\lfloor (k+j)/2 \rfloor}{\lfloor (k-j)/2 \rfloor},$$

$0 \leq j \leq k$. *The roots of* $m_k(x)$ *are the Galois conjugates of the cyclotomic units* $(-1)^{k-1} \lim_{\tau \to 0} r_{d,1}(\tau)$, *where* d *ranges over the divisors of* ℓ, $d > 1$.

Corollary 1.1. *The minimal polynomial for the Gauss periods of degree* 2 *for odd* $\ell \geq 3$ *has coefficients of* x^j, $0 \leq j \leq k$, *given by*

$$d_{k-j} = \begin{cases} (-1)^{\lfloor \frac{k+1}{2} \rfloor} c_{k-j} & j \text{ odd} \\ (-1)^{\lfloor \frac{k}{2} \rfloor} c_{k-j} & j \text{ even} \end{cases}$$

The structure of these modular unit groups and their associated cuspidal divisor class groups are further characterized in [6], where in particular we provide a cuspidal divisor class number formula given in terms of products of L-functions.

2. Rogers-Ramanujan

The Rogers-Ramanujan continued fraction, defined by

$$r(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \ddots}}}} \tag{2.1}$$

where $q = q(\tau) := e^{2\pi i \tau}$, is convergent for τ in the upper half complex plane

$\mathcal{H}$, and has been an object of extensive study due to a wealth of associated analytic properties, q-series identities, and combinatorial interpretations (see [2], for example). Here instead, we present $r(\tau)$ as an example of a modular unit, and define the modular units $r_{\ell,j}, 1 \leq j \leq (\ell-3)/2$ in § 3 as higher level analogues to the Rogers-Ramanujan function $r(\tau)$.

Many of the fundamental properties associated to $r(\tau)$ rely upon the fact that one may regard $r(\tau)$ as arising from the q-recurrence

$$R(z) = R(zq) + zqR(zq^2). \tag{2.2}$$

That is, we may write

$$r(\tau) = q^{1/5}R(q)/R(1). \tag{2.3}$$

The q-recurrence (2.2) has a known analytic solution [23] given by

$$R(z) = \sum_{n\geq 0} z^n q^{n^2} (q;q)_n^{-1}. \tag{2.4}$$

Here we use the $q-$Pochhammer symbol defined by

$$(a;q)_k = \begin{cases} 1 & k=0 \\ (1-a)(1-aq)\cdots(1-aq^{k-1}) & k=1,2,\ldots \\ [(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{-k})]^{-1} & k=-1,-2,\ldots \\ (1-a)(1-aq)(1-aq^2)(1-aq^3)\cdots & k=\infty. \end{cases}$$

A celebrated result associated to $r(\tau)$ are the Rogers-Ramanujan identities,

$$R(1)^{-1} = \prod_{n\geq 1}(1-q^{5n-1})(1-q^{5n-4}) \tag{2.5}$$

$$R(q)^{-1} = \prod_{n\geq 1}(1-q^{5n-2})(1-q^{5n-3}), \tag{2.6}$$

which give an infinite product representation for the series (2.4) evaluated at $z=1$ and $z=q$. Various proofs have emerged in the literature for the identities (2.5) and (2.6), originally due to Rogers and (independently) Ramanujan, whose proofs rely upon the use of Theta functions [23].[2] The identities may also be interpreted combinatorially (see for example [1]) in terms of number partitions. For example (2.5) may be read as saying "the number of ways to partition a number into parts congruent to 1 and 4 mod 5 is equal the number of partitions into parts of minimal difference 2."

By (2.3), (2.4), (2.5) and (2.6), one obtains a q-product expansion for $r(\tau)$. With this, one may show that $r(\tau)$ is in fact a modular function on $\Gamma(5)$, and also a modular unit. We also observe a parallel role played

[2] See section 4.

by a specialization of $r(\tau)$ within the number fields. By definition, the specialization of $r(\tau)$ in the cusp 0 is easily observed:

$$\lim_{\tau\to 0} r(\tau) = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+\ \cdots} = (-1+\sqrt{5})/2. \tag{2.7}$$

Rewriting the special value (2.7) another way, we find

$$(-1+\sqrt{5})/2 = (\zeta_5^2 - \zeta_5^{-2})(\zeta_5 - \zeta_5^{-1})^{-1}, \tag{2.8}$$

which we recognize as a cyclotomic unit. This special value is real, and in fact generates the group of cyclotomic units in $\mathbb{Q}(\zeta_5)^+$, where we let K^+ denote the maximal real subfield of a given field K. We also note this specialization may be used to generate the field $\mathbb{Q}(\zeta_5)^+$, that is, $\mathbb{Q}(\zeta_5)^+ = \mathbb{Q}(\lim_{\tau\to 0} r(\tau))$. Rephrasing, we have a modular unit arising from a q-recurrence that generates a modular function field, and whose specialization in the cusp 0 generates a real cyclotomic field, as well as its cyclotomic unit group. In what follows we consider the fields $\mathcal{F}_\ell$, $\ell \geq 5$, as well as specializations at other τ in $\mathcal{H}$, and provide more general proofs of these facts.

3. The modular unit groups U_ℓ^C

To define higher order modular unit groups analogous to $U_5^C = \langle r(\tau)\rangle$ generated by the Rogers-Ramanujan continued fraction $r(\tau)$, we seek other appropriate modular functions on the curves $X(N)$. Due to the fact that $r(\tau)$ has a continued fraction expansion and may be viewed as originating from a certain q-recurrence, one might hope that if analogous functions exist, that they exhibit similar traits. We indeed form such families for curves $X(N)$ of higher genus $g > 0$ in what follows, yet will see that these functions do not possess a continued fraction expansion but rather an expression that may be viewed as a generalization of a continued fraction expansion. To describe this, we begin with the observation of Sylvester that a continued fraction may be expressed as the limiting value of ratios of n^{th} order determinants. That is,

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} = \begin{vmatrix} a_0 & b_1 & 0 & 0 & \cdots \\ -1 & a_1 & b_2 & 0 & \cdots \\ 0 & -1 & a_2 & b_3 & \cdots \\ 0 & 0 & -1 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & 0 & \cdots \\ 0 & -1 & a_2 & b_3 & \cdots \\ 0 & 0 & -1 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}^{-1}$$

where the successive ratios of order n determinants are the partial convergents of the continued fraction. One finds that the Rogers-Ramanujan continued fraction may be expressed as

$$
r(\tau) = q^{1/5} \begin{vmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & q^2 & 0 & \cdots \\ 0 & -1 & 1 & q^3 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \begin{vmatrix} 1 & q & 0 & 0 & \cdots \\ -1 & 1 & q^2 & 0 & \cdots \\ 0 & -1 & 1 & q^3 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}^{-1}. \tag{3.1}
$$

We generalize this definition, and let $A^{(n)}$ be the order n determinant with entries $a_{i,j}$, $1 \le i, j \le n$, with $B^{(n)}$ defined similarly. Provided the limit exists, we define the ratio of two infinite determinants by

$$
\lim_{n\to\infty} A^{(n)}/B^{(n)} := \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \cdots \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \begin{vmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & \cdots \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & \cdots \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & \cdots \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}^{-1}. \tag{3.2}
$$

With hopes of constructing analogous modular functions to (3.1) of higher level, we form the infinite determinant $S_k(z)$ of width $k+1$

$$
S_k(z) := \begin{vmatrix} s_{k,0}(z) & s_{k,1}(z) & s_{k,2}(z) & \cdots & s_{k,k-1}(z) & 0 & 0 & 0 & \cdots \\ -1 & s_{k,0}(zq) & s_{k,1}(zq) & s_{k,2}(zq) & \cdots & s_{k,k-1}(zq) & 0 & 0 & \cdots \\ 0 & -1 & s_{k,0}(zq^2) & s_{k,1}(zq^2) & s_{k,2}(zq^2) & \cdots & s_{k,k-1}(zq^2) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix}. \tag{3.3}
$$

One may deduce from (3.3) the higher order q-recurrences satisfied by the functions $S_k(z)$

$$
\sum_{m=0}^{k} s_{k,m-1}(z) S_k(zq^m) = 0 \tag{3.4}
$$

where we let $s_{k,-1}(z) = -1$. We must define the functions $s_{k,j}(z)$ appropriately so that (3.4) yields an analytic solution $S_k(z)$.

We let

$$s_{k,2n}(z) := (-1)^n z^{nk} q^{\delta(k,n)} (q;q)_n^{-2} \times \sum_{j=0}^{\lfloor \frac{k+1}{2} \rfloor - n - 1} z^j q^{j(n+1)} (q^{j+1};q)_n (q^{\lfloor \frac{k+1}{2} \rfloor - (j+n)};q)_n \tag{3.5}$$

$$s_{k,2n-1}(z) := (-1)^{n+1} z^{nk - \lfloor \frac{k}{2} \rfloor} q^{\delta'(k,n)} (q;q)_{n-1}^{-2} (1-q^n)^{-1} \times \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor - n} z^j q^{jn} (q^{j+1};q)_n (q^{\lfloor \frac{k}{2} \rfloor - (j+n)+1};q)_{n-1} \tag{3.6}$$

for $n \geq 0$ in (3.5), and $n \geq 1$ in (3.6), where

$$\delta(k,n) = (2k+1)\frac{n^2+n}{2} - n\left\lfloor \frac{k+1}{2} \right\rfloor$$

$$\delta'(k,n) = (2k+1)\frac{n^2-n}{2} + n\left\lfloor \frac{k+1}{2} \right\rfloor,$$ [3]

and

$$\lfloor \alpha \rfloor = \max_{n\in\mathbb{Z}} \{n \leq \alpha\}$$

$$\lceil \alpha \rceil = \min_{n\in\mathbb{Z}} \{n \geq \alpha\}.$$

As given in (3.5) and (3.6), the functions $s_{k,m}(z)$ first appear in the work of Selberg [26], and in the case $k=2$, the q-recurrence (3.4) reduces to (2.2), with $S_2(z) = R(z)$. With this choice of $s_{k,j}$, an analytic solution to the q-recurrence (3.4) exists, and is given in [26] by

$$S_k(z) = \sum_{n=0}^{\infty} (-1)^n z^{kn} q^{(2k+1)\frac{n^2+n}{2} - kn} (1 - z^k q^{(2n+1)k}) \frac{(zq;q)_n}{(q;q)_n} \times \prod_{m\geq 1} (1-zq^m)^{-1}. \tag{3.7}$$

Selberg uses these functions to give certain identities analogous to the Rogers-Ramanujan identities (2.5) and (2.6), and also to give various q-continued fraction identities. In contrast to their use in [26], we use the functions $s_{k,j}(z)$ and $S_k(z)$ to define groups of modular units U_N^C. To define the modular units $r_{\ell,j}$, we first recall the Galois action on the modular function fields $\mathcal{F}_N$.

The natural action of the group $\Gamma(1) \subset \mathrm{M}_2(\mathbb{Z})$ on the fields $\mathcal{F}_N$ given by

$$\gamma \cdot f(\tau) = f(\gamma\tau), \tag{3.8}$$

where $f = f(\tau) \in \mathcal{F}_N$, may be extended to the group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ as follows. Given $d \in (\mathbb{Z}/N\mathbb{Z})^*$, let

$$\gamma_d := \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right) \tag{3.9}$$

act on $\mathbb{Q}(\zeta_N)$ by

$$\gamma_d \cdot \zeta_N = \zeta_N^d.$$

If $f \in \mathcal{F}_N$ has q-series expansion given by $f(\tau) = \sum_{n=m}^{\infty} a_n q^{n/N}$, $a_n \in \mathbb{Q}(\zeta_N)$, the action of γ_d extends to $\mathcal{F}_N$ by

$$\gamma_d \cdot f(\tau) = \sum_{n=m}^{\infty} (\gamma_d \cdot a_n) q^{n/N}. \tag{3.10}$$

The matrices γ_d, $d \in (\mathbb{Z}/N\mathbb{Z})^*$, together with $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$, generate $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, and the group actions given by (3.8) and (3.10) define a Galois action on the modular function fields, described by the following exact sequence

$$1 \longrightarrow G_N \cdot \{\pm 1\} \longrightarrow G \longrightarrow \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \longrightarrow 1 \tag{3.11}$$

where

$$G = \prod_p \mathrm{GL}_2(\mathbb{Z}_p) \cdot G_\infty^+ \tag{3.12}$$

$$G_N = \{(x) \in G/G_\infty^+ \mid x_p \equiv 1 \mod N \cdot \mathrm{M}_2(\mathbb{Z}_p)\}. \tag{3.13}$$

The product defining G is taken over primes p, and $\mathbb{Z}_p$ denotes the ring of integers in the completion $\mathbb{Q}_p$ of the field $\mathbb{Q}$ at p. From (3.11), one finds the isomorphism

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \overset{\pi}{\cong} \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}. \tag{3.14}$$

We first define using (3.7) the functions $r_{\ell,1}$ and $r_{\ell,2}$.

Definition. For $\ell = 2k+1 \geq 5$, let

$$r_{\ell,1}(\tau) := (-1)^{k-1} q^{\frac{-k(k-1)}{2\ell}} \frac{S_k(1)}{S_k(q)} \tag{3.15}$$

and for $\ell = 2k+1 > 5$, let

$$r_{\ell,2}(\tau) := (-1)^{k-2} q^{\frac{-(k+1)(k-2)}{2\ell}} \frac{S_k(1) - q^{k-1} S_k(q^2)}{S_k(q)}. \tag{3.16}$$

For $m \in (\mathbb{Z}/N\mathbb{Z})^*$, we let

$$\sigma_m := \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \pi\big(\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)\big) \tag{3.17}$$

and for $\ell = 2k+1$, let

$$\Psi_{\ell,k} := -q^{\frac{(3\ell-4)(\ell-3)}{24\ell}} \frac{\eta(\tau)}{\eta(\ell\tau)} S_k(q), \tag{3.18}$$

$$\Psi_1 := -e(-k^2/\ell)\Psi_{\ell,k}. \tag{3.19}$$

We define for $1 \le j \le (\ell-3)/2$ and $\gcd(k+1-j, \ell) = 1$ the functions

$$r_{\ell,j}(\tau) := \sigma_{k+1-j}\big(i\Psi_{\ell,k}\big)/i\Psi_{\ell,k}, \tag{3.20}$$

where the Dedekind η-function $\eta(\tau) = \Delta^{1/24}(\tau)$ is the 24^{th} root of the Discriminant function, and is given by

$$\eta(\tau) = q^{1/24} \prod_{n\ge 1} (1-q^n). \tag{3.21}$$

A consistent definition for j such that $\gcd(k+1-j, \ell) > 1$ appears in the following sections. We will show in §5 that definitions (3.15) and (3.16) coincide with (3.20) for $j = 1, 2$.

4. Theta constants

To prove Theorems 1.1 and 1.2 we will make use of the theory of the theta constants. A theta characteristic is a vector $\chi = \left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right] \in \mathbb{R}^2$. Two characteristics are said to be equivalent if their difference is in $\mathbb{Z}^2$, and the space of characteristic classes is defined as $\mathbb{R}^2$ modulo this equivalence relation. Given a matrix $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma(1)/\{\pm 1\}$, a right group action on the space of characteristic classes is defined by

$$\chi\gamma = \gamma^t \chi + \left[\begin{smallmatrix} -ac\\ bd\end{smallmatrix}\right] \tag{4.1}$$

The theta constant with characteristic $\chi = \left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right] \in \mathbb{R}^2$ is defined by

$$\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\tau) = \sum_{n\in\mathbb{Z}} e\Big(\frac{1}{2}\Big(n+\frac{\epsilon}{2}\Big)^2\tau + \Big(n+\frac{\epsilon}{2}\Big)\frac{\epsilon'}{2}\Big).$$

This function converges for $\tau \in \mathcal{H}$, and satisfies the transformation rule

$$\theta\left[\chi\right](\gamma\tau) = \kappa_{\chi,\gamma}(c\tau+d)^{1/2}\theta\left[\chi\gamma\right](\tau) \tag{4.2}$$

for $\gamma \in \Gamma(1)$, and

$$\kappa_{\chi,\gamma} = e\Big(-\frac{1}{4}(a\epsilon + c\epsilon')bd - \frac{1}{8}(ab\epsilon^2 + cd\epsilon'^2 + 2bc\epsilon\epsilon')\Big)\kappa_\gamma$$

where κ_γ is an eighth root of unity depending only on the matrix γ. As a special case of (4.2), one finds for $r, s \in \mathbb{Z}$,

$$\theta\left[\begin{smallmatrix}\pm\epsilon\ +2r\\ \pm\epsilon'+2s\end{smallmatrix}\right](\tau) = e\Big(\pm\frac{\epsilon s}{2}\Big)\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\tau). \tag{4.3}$$

The theta constants also satisfy the product identity

(4.4)

$$\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\tau) = e(\tfrac{\epsilon\epsilon'}{4})q^{\epsilon^2/8}\prod_{n\geq 1}(1-q^n)(1+e(\tfrac{\epsilon'}{2})q^{n-\frac{1+\epsilon}{2}})(1+e(\tfrac{-\epsilon'}{2})q^{n-\frac{1-\epsilon}{2}}),$$

which can be derived from the Jacobi triple product identity [5]

$$\sum_{n=-\infty}^{\infty} q^{n^2}z^{2n} = \prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z^2)(1+q^{2n-1}z^{-2}). \tag{4.5}$$

To prove statement $i.$ of Theorem 1.1, we begin by applying the identity

$$\sum_{n\in\mathbb{Z}} e\left(\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2\tau + \left(n+\frac{\epsilon}{2}\right)\frac{\epsilon'}{2}\right)$$
$$= e\left(\frac{\epsilon\epsilon'}{4}\right)q^{\epsilon^2/8}\prod_{n\geq 1}(1-q^n)(1+e(\epsilon'/2)q^{n-\frac{1+\epsilon}{2}})(1+e(-\epsilon'/2)q^{n-\frac{1-\epsilon}{2}}) \tag{4.6}$$

which may also be derived from (4.5). Thus by (3.7) and (4.6) one finds

$$S_k(1) = \prod_{n\geq 1}\frac{(1-q^{\ell n-k})(1-q^{\ell n-(k+1)})(1-q^{\ell n})}{(1-q^n)} \tag{4.7}$$

$$S_k(q) = \prod_{n\geq 1}\frac{(1-q^{\ell n-1})(1-q^{\ell n-(\ell-1)})(1-q^{\ell n})}{(1-q^n)} \tag{4.8}$$

$$S_k(1) - q^{k-1}S_k(q^2) = \prod_{n\geq 1}\frac{(1-q^{\ell n-(k-1)})(1-q^{\ell n-(k+2)})(1-q^{\ell n})}{(1-q^n)}. \tag{4.9}$$

Thus, by (3.15) and (3.16), (4.7), (4.8), and (4.9),

$$r_{\ell,1} = (-1)^{k-1}q^{-k(k-1)/2\ell}\prod_{n\geq 1}\frac{(1-q^{\ell n-k})(1-q^{\ell n-(k+1)})}{(1-q^{\ell n-1})(1-q^{\ell n-(\ell-1)})} \tag{4.10}$$

$$r_{\ell,2} = (-1)^{k-2}q^{-(k+1)(k-2)/2\ell}\prod_{n\geq 1}\frac{(1-q^{\ell n-(k-1)})(1-q^{\ell n-(k+2)})}{(1-q^{\ell n-1})(1-q^{\ell n-(\ell-1)})}. \tag{4.11}$$

Using (4.10) and (4.11) we apply (4.4) and (4.2), and will see in §5 using (3.20) the following expression for the modular units $r_{\ell,m}(\tau)$, where $1 \leq m \leq (\ell-3)/2$.

Proposition 4.1. *For* $\ell = 2k+1 \geq 5$, $1 \leq m \leq (\ell-3)/2$,

$$r_{\ell,m}(\tau) = (-1)^{k-m}e(\tfrac{k-m}{2\ell})\frac{\theta\left[\begin{smallmatrix}\frac{2m-1}{\ell}\\ 1\end{smallmatrix}\right](\ell\tau)}{\theta\left[\begin{smallmatrix}\frac{\ell-2}{\ell}\\ 1\end{smallmatrix}\right](\ell\tau)}. \tag{4.12}$$

We will use the following Lemma.

Lemma 4.1. *For any odd $\ell\epsilon$, ϵ' and ℓ, and any $\gamma \in \Gamma(\ell)$, the theta constant $\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\ell\tau)$ satisfies the transformation*

$$\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\ell\cdot\gamma\tau) = \nu_{\chi,\gamma}(cz+d)^{1/2}\ \theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\ell\tau)$$

where

$$\nu_{\chi,\gamma} = e(-\epsilon d(a-1)(b\ell+\epsilon')/4)e(-\epsilon^2(b\ell(a-1))/8)e(-\epsilon' d(2bc+c\epsilon'/\ell)/8).$$

Proof. (Lemma 4.1) We compute

$$\theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\ell\cdot\gamma\tau) = \theta\left[\begin{smallmatrix}\epsilon\\ \epsilon'\end{smallmatrix}\right](\tilde{\gamma}\circ\ell\tau)$$

where $\tilde{\gamma} = \left(\begin{smallmatrix}a & b\ell\\ c/\ell & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and find

$$\begin{aligned}\chi\tilde{\gamma} &= \tilde{\gamma}^t\chi + \left[\begin{smallmatrix}-ac/\ell\\ bd\ell\end{smallmatrix}\right]\\ &= \left[\begin{smallmatrix}a\epsilon + c\epsilon'/\ell - ac/\ell\\ b\ell\epsilon + d\epsilon' + bd\ell\end{smallmatrix}\right].\end{aligned}$$

For ease of notation, let $\gamma(\epsilon) = a\epsilon + c\epsilon'/\ell - ac/\ell$ and $\gamma(\epsilon') = b\ell\epsilon + d\epsilon' + bd\ell$.

If $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix}1+A\ell & B\ell\\ C\ell & 1+D\ell\end{smallmatrix}\right)$, we find

$$\begin{aligned}\gamma(\epsilon) &= \epsilon + A\ell\epsilon + C\epsilon' - C(1+A\ell)\\ \gamma(\epsilon') &= \epsilon' + D\ell\epsilon' + B\ell^2\epsilon + B\ell^2(1+D\ell)\end{aligned}$$

If A is odd, then C must be odd, as $\det(\gamma) = 1$. In this case, $\gamma(\epsilon) \equiv \epsilon \bmod 2$. This also holds if A is even, regardless of the parity of C. Similarly, if D is odd, then B is odd, so that $\gamma(\epsilon') \equiv \epsilon' \bmod 2$. Again we find that this also holds if D is even. By (4.2) and (4.3) we have

$$\begin{aligned}\theta[\chi](\tilde{\gamma}\circ\ell\tau) &= \kappa_{\chi,\tilde{\gamma}}(\tfrac{c}{\ell}\cdot\ell\tau + d)^{1/2}\theta[\chi\tilde{\gamma}](\ell\tau)\\ &= \kappa_{\chi,\tilde{\gamma}}e(\tfrac{\epsilon}{4}(b\ell\epsilon + (d-1)\epsilon' + bd\ell))(c\tau+d)^{1/2}\theta[\chi](\ell\tau).\end{aligned}$$

Combining the constants gives the expression for $\nu_{\chi,\gamma}$ and completes the proof. □

Proof. (Theorem 1.1 *i.*)

To show that the functions $r_{\ell,j}$ are modular units of level ℓ, we see by (4.10), (4.11) and (3.20) that the $r_{\ell,j}$ are holomorphic on $\mathcal{H}$. Thus it suffices to show the functions transform correctly under $\Gamma(\ell)$, and that they in fact lie in U_ℓ. To show transformation, by Lemma 1 and (4.12) it suffices to show that $\nu_{\chi_1,\gamma} = \nu_{\chi_2,\gamma}$, where

$$\chi_1 = \begin{bmatrix} \epsilon_1 \\ \epsilon' \end{bmatrix} = \begin{bmatrix} \frac{2m-1}{\ell} \\ 1 \end{bmatrix}$$

$$\chi_2 = \begin{bmatrix} \epsilon_2 \\ \epsilon' \end{bmatrix} = \begin{bmatrix} \frac{\ell-2}{\ell} \\ 1 \end{bmatrix}.$$

We have

$$\begin{aligned} \frac{\nu_{\chi_1,\gamma}}{\nu_{\chi_2,\gamma}} &= e(\tfrac{d(a-1)}{4}(\epsilon_2 - \epsilon_1)(1+b\ell))e\Big(\tfrac{b\ell(a-1)}{8}(\epsilon_2^2 - \epsilon_1^2)\Big) \\ &= e\Big(\tfrac{d(a-1)}{2\ell}(k-m)(1+b\ell)\Big)e\Big(\tfrac{b(a-1)}{2\ell}(k-m)(k+m+1)\Big) \\ &= e\Big(\tfrac{dA}{2}(k-m)(1+b\ell)\Big)e\Big(\tfrac{B(a-1)}{2}(k-m)(k+m+1)\Big) \\ &= e\Big(\tfrac{dA}{2}(k-m)(1+b\ell)\Big). \end{aligned} \tag{4.13}$$

The last equality follows from the fact that $(k-m)$ and $(k+m+1)$ have opposite parity. Finally, if b is odd, then the expression in (4.13) simplifies to 1. If b is even, then $ad = (1+\ell A)d$ is odd, so that A is even, and again the expression (4.13) is equal to 1. By examining (4.11), (4.11) and (3.20), one finds that the functions $r_{\ell,j}$ have divisors supported on the cusps of $\Gamma(\ell)$. This proves Theorem 1.1 *i*. □

Proof. (Theorem 1.2). For the involution $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we have by (4.2)

$$\theta\big[\begin{smallmatrix} \epsilon \\ \epsilon' \end{smallmatrix}\big](-1/\tau) = e\left(\frac{\epsilon\epsilon'}{4}\right)\kappa_\gamma \tau^{1/2}\theta\big[\begin{smallmatrix} \epsilon' \\ -\epsilon \end{smallmatrix}\big](\tau)$$

so that

$$\begin{aligned} \lim_{\tau\to 0} r_{\ell,m}(\tau) &= \lim_{\tau\to\infty} r_{\ell,m}(-1/\tau) \\ &= \lim_{\tau\to\infty} (-1)^{k-m} e(\tfrac{k-m}{2\ell}) \frac{\theta\begin{bmatrix} \frac{2m-1}{\ell} \\ 1 \end{bmatrix}(-1/\ell\tau)}{\theta\begin{bmatrix} \frac{\ell-2}{\ell} \\ 1 \end{bmatrix}(-1/\ell\tau)} \\ &= \lim_{\tau\to\infty} (-1)^{k-m} \frac{\theta\begin{bmatrix} 1 \\ \frac{1-2m}{\ell} \end{bmatrix}(\ell\tau)}{\theta\begin{bmatrix} 1 \\ \frac{2-\ell}{\ell} \end{bmatrix}(\ell\tau)}. \end{aligned}$$

We use the fact that

$$\theta\left[\begin{smallmatrix}1\\ \epsilon'\end{smallmatrix}\right](\tau) = q^{1/8}e(\epsilon'/4)\sum_{n\in\mathbb{Z}} e((n^2+n)\tau/2 + n\epsilon'/2)$$

$$= q^{1/8}(e(\epsilon'/4) + e(-\epsilon'/4) + \mathrm{O}(|q|)) \tag{4.14}$$

as $\tau \to \infty$, to conclude

$$\lim_{\tau\to 0} r_{\ell,m}(\tau) = (-1)^{k-m}\frac{\zeta_\ell^{\frac{2m-1}{4}} + \zeta_\ell^{\frac{1-2m}{4}}}{\zeta_\ell^{\frac{\ell-2}{4}} + \zeta_\ell^{\frac{2-\ell}{4}}}$$

$$= (-1)^{k-m}\frac{\zeta_\ell^{\frac{k+1-m}{2}} - \zeta_\ell^{-\frac{k+1-m}{2}}}{\zeta_\ell^{\frac{1}{2}} - \zeta_\ell^{-\frac{1}{2}}}$$

$$= (-1)^{k-m}\zeta_\ell^{\frac{m-k}{2}}\frac{\zeta_\ell^{k+1-m} - 1}{\zeta_\ell - 1}$$

$$= (-1)^{k-m}\zeta_\ell^{\frac{1-v}{2}}\frac{\zeta_\ell^{v} - 1}{\zeta_\ell - 1} \tag{4.15}$$

where $v = k+1-m$. To conclude the proof, we state two well known results regarding the description of the cyclotomic unit groups [29].

Lemma 4.2. *For $N = p^j$, $p \geq 5$ prime, $j \geq 1$,*

$$E^C_{\mathbb{Q}(\zeta_N)} = \langle\{\zeta_N, E^C_{\mathbb{Q}(\zeta_N)^+}\}\rangle.$$

Lemma 4.3. *For $N = p^j$, p prime, $j \geq 1$,*

$$E^C_{\mathbb{Q}(\zeta_N)^+} = \Big\langle\Big\{ -1,\ \zeta_N^{(1-m)/2}\frac{1-\zeta_N^m}{1-\zeta_N} \mid 1 < m < N/2, (m,p) = 1\Big\}\Big\rangle.$$

As m ranges over integers $\{1, 2, 3, \ldots, \frac{\ell-1}{2} - 1\}$, we note that v ranges over integers $\{2, 3, \ldots, \frac{\ell-1}{2}\}$. With v in this range, by Lemma 4.3 we see that the set of specializations $\lim_{\tau\to 0} S_\ell^C$ may be used to generate the group $E^C_{\mathbb{Q}(\zeta_\ell)^+}$ of cyclotomic units in $\mathbb{Q}(\zeta_\ell^+)$. When ℓ is not a prime power, there may be multiplicative dependence between the elements in $\lim_{\tau\to 0} S_\ell^C$, although these primitive cyclotomic units in $\mathbb{Q}(\zeta_\ell)^+$ may be used to define a set of multiplicatively independent units as given by Ramachandra [21]. This proves Theorem 1.2.. □

To prove statement *ii.* of Theorem 1.1, we turn to a discussion of the Siegel functions.

5. Siegel functions

The Siegel functions are constructed using the Klein forms, $\mathfrak{t}_L(z)$, which are defined by a lattice $L \subseteq \mathbb{C}$, and are functions of $z \in \mathbb{C}$. Equivalently, we may write $\mathfrak{t}_L(z) = \mathfrak{t}_a(\tau)$ if $L = \mathbb{Z}\tau + \mathbb{Z}$, and $z = a_1\tau + a_2 \in \mathbb{C}$, $a_1, a_2 \in \mathbb{R}$. With the latter notation, the Klein forms are defined by

$$\mathfrak{t}_a(\tau) = e^{-\eta_a(\tau)a\cdot(\tau,1)/2}\sigma_a(\tau), \tag{5.1}$$

where σ_a and η_a are the classical Weierstrass functions. Using known properties of the Weierstrass σ_a and η_a functions, one can verify that the Klein forms satisfy the following properties

$$\mathfrak{t}_{\lambda L}(\lambda z) = \lambda \mathfrak{t}_L(z) \tag{5.2}$$

$$(c\tau + d)\mathfrak{t}_a(\gamma\tau) = \mathfrak{t}_{a\gamma}(\tau) \tag{5.3}$$

$$\mathfrak{t}_{a+b}(\tau) = \epsilon(a,b)\mathfrak{t}_a(\tau). \tag{5.4}$$

In (5.2)-(5.4), $\lambda \in \mathbb{C}^*$, $\gamma \in \Gamma(1), b = (b_1, b_2) \in \mathbb{Z}^2$, and the constant $\epsilon(a,b)$ is given by

$$\epsilon(a,b) = (-1)^{b_1b_2+b_1+b_2}e((b_2a_1 - b_1a_2)/2).$$

We note in particular that (5.2) implies

$$\mathfrak{t}_{-a}(\tau) = -\mathfrak{t}_a(\tau).$$

The Klein forms are used to define the Siegel functions, given by

$$g_a(\tau) = \mathfrak{t}_a(\tau)\eta(\tau)^2, \tag{5.5}$$

where $\eta(\tau)$ is the Dedekind η-function with q-development given by (3.21). We point out the following transformation law.

Lemma 5.1. *Let $a = (a_1, a_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$. Then*

$$g_{(a_1,a_2)}(-1/\tau) = -ig_{(a_2,-a_1)}(\tau).$$

Proof. Using the definition (5.5) of the Siegel function g_a, the lemma follows from the transformation properties

$$\eta^2(-1/\tau) = -i\tau\eta^2(\tau) \tag{5.6}$$

$$\tau\mathfrak{t}_{(a_1,a_2)}(-1/\tau) = \mathfrak{t}_{(a_2,-a_1)}(\tau). \tag{5.7}$$

□

A major result of Kubert and Lang in their development of the theory of the modular units is the following explicit characterization of the modular units of prime power level ℓ.

Theorem 5.1. *(Kubert, Lang) Let $\ell = p^n$, where p is prime, $p \neq 2, 3$, $n \in \mathbb{N}$. Then the modular units of level ℓ (modulo constants) consist of products*

$$\prod_a g_a^{m(a)}$$

where $a = (\frac{a_1}{\ell}, \frac{a_2}{\ell}) \in \frac{1}{\ell}\mathbb{Z}^2$, $a \notin \mathbb{Z}^2$, and the exponents $m(a) \in \mathbb{Z}$ satisfy the quadratic relations

$$\sum_a m(a)a_1^2 \equiv \sum_a m(a)a_2^2 \equiv \sum_a m(a)a_1a_2 \equiv 0 \bmod \ell,$$

and the condition $\sum_a m(a) \equiv 0 \bmod 12$.

Using the product expansions for the η and σ functions, one has the following product formula for the Siegel functions:

$$g_a(\tau) = -q^{\frac{1}{2}\mathbf{B}_2(\frac{a_1}{\ell})} e\left(\frac{a_2}{\ell}\left(\frac{a_1}{\ell} - 1\right)/2\right)\left(1 - q^{\frac{a_1}{\ell}} e(\tfrac{a_2}{\ell})\right) \times \prod_{n=1}^{\infty}\left(1 - q^{n+\frac{a_1}{\ell}} e(\tfrac{a_2}{\ell})\right)\left(1 - q^{n-\frac{a_1}{\ell}} e(-\tfrac{a_2}{\ell})\right) \tag{5.8}$$

where $\mathbf{B}_2(z)$ is the second Bernoulli polynomial defined by

$$\mathbf{B}_2(z) = z^2 - z + \frac{1}{6}.$$

One may verify using (5.3), (5.8), and the Galois action described in § 3, that the Galois group $\mathrm{Gal}(\mathcal{F}_\ell/\mathcal{F}_1)$ acts on the Siegel functions by multiplication on the indices. That is, for $\beta \in \mathrm{Gal}(\mathcal{F}_\ell/\mathcal{F}_1)$,

$$\beta \cdot g_a = g_{a\cdot\beta}. \tag{5.9}$$

We emphasize the following corollary.

Corollary 5.1. *For $\ell = 2k+1$ prime, the function Ψ_1 as defined in (3.19) is a modular unit of level 12ℓ, with order at ∞ given by*

$$\mathrm{ord}_\infty(\Psi_1) = \frac{(\ell-2)(\ell-3)}{12\ell} - \frac{1}{12}.$$

Further, the function Ψ_1^{12} is a modular unit of level ℓ.

Proof. Using the definition of $\Psi_{\ell,k}$ given in (3.18), the product expansions (3.21), (4.8), and (5.8), one may verify that

$$\Psi_1 = \prod_{s=0}^{\ell-1} g_{(1/\ell, s/\ell)} \tag{5.10}$$

$$= -q^{\frac{\ell}{2}B_2(1/\ell)} e(-k^2/\ell) \prod_{n=1}^{\infty}(1 - q^{n\ell-(\ell-1)})(1 - q^{n\ell-1}). \tag{5.11}$$

Using (5.10), the modularity of Ψ_1 and Ψ_1^{12} follows as an immediate consequence of Theorem 5.1. The order of Ψ_1 at ∞ is seen in (5.11) to be $\ell B_2(1/\ell)/2$, which is equal to $\frac{(\ell-2)(\ell-3)}{12\ell} - \frac{1}{12}$. □

We now offer the following factorization of the functions $r_{\ell,j}$ into a product of Siegel functions.

Proposition 5.1. *For* $\ell = 2k+1 \geq 5$ *and* $2 \leq m \leq k$,

$$r_{\ell,k+1-m} = (-1)^{(m-1)} e\left(-\frac{k(m-1)}{2\ell}\right) \prod_{s=0}^{\ell-1} g_{(m/\ell,s/\ell)}/g_{(1/\ell,s/\ell)} \tag{5.12}$$

$$= (-1)^{m-1} \prod_{s=-k}^{k} \left(\frac{g_{(m/\ell,s/\ell)}}{g_{(1/\ell,s/\ell)}}\right). \tag{5.13}$$

Proof. Using Lemma 5.1, we first compute

$$i\Psi_{\ell,k}(-1/\tau) = -e(k^2/\ell) i\Psi_1(-1/\tau) \tag{5.14}$$

$$= (-1)^{k+1} e(k^2/\ell) \prod_{s=0}^{\ell-1} g_{(s/\ell,-1/\ell)}, \tag{5.15}$$

and thus

$$\gamma_m(i\Psi_{\ell,k}(-1/\tau)) = (-1)^m e\left(-\frac{km}{2\ell}\right) \prod_{s=0}^{\ell-1} g_{(s/\ell,-m/\ell)},$$

where γ_m is as defined in (3.9). We use the fact that $\sigma_m = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\gamma_m\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and apply Lemma 5.1 once more to conclude for $r_{\ell,k+1-m}$ as defined in (3.20),

$$\begin{aligned} r_{\ell,k+1-m}(\tau) &= (-1)^{m-1} e\left(-\frac{k(m-1)}{2\ell}\right) \prod_{s=0}^{\ell-1} \left(\frac{g_{(m/\ell,s/\ell)}}{g_{(1/\ell,s/\ell)}}\right) \\ &= (-1)^{m-1} \prod_{s=-k}^{k} \left(\frac{g_{(m/\ell,s/\ell)}}{g_{(1/\ell,s/\ell)}}\right), \end{aligned}$$

where the last equality can be derived by using (5.4). □

Remark. We remark that the action of σ_m is well defined mod ℓ, i.e. for any integers t and m relatively prime to ℓ, $\sigma_t(r_{k+1-m}) = \sigma_{t^*}(r_{k+1-m})$, where $t^* \equiv t$ in $(\mathbb{Z}/\ell Z)^*/\{\pm 1\}$. One may see this by first observing that the $q^{1/12\ell}$-expansion at $i\infty$ of $i\Psi_{\ell,k}(-1/\tau)$ is given by

(5.16)
$$i\Psi_{\ell,k}(-1/\tau)=(e(k/\ell)-e(-k/\ell))q^{\frac{1}{12\ell}}\prod_{n=1}^{\infty}(1-q^{\frac{n}{\ell}}e(-1/\ell))(1-q^{\frac{n}{\ell}}e(1/\ell)).$$

By (3.20), one has $\sigma_t(r_{k+1-m}) = r_{k+1-tm}/r_{k+1-t}$ (for t and m relatively prime to ℓ). Combining this with the fact that $r_{k+1-m} = -r_{k+1+m}$ for any integer m such that $m \not\equiv 0 \mod \ell$, proves the assertion that $\sigma_t(r_{k+1-m}) = \sigma_{t^*}(r_{k+1-m})$.

Proof. (Proposition 4.1) Proposition 4.1 now follows from Proposition 5.1, (5.8) and (4.4). □

We now proceed with the proof of Theorem 1.1 *ii*. As before, let $\mathcal{F}_\ell$ be the function field of the curve $X(\ell)$ over $\mathbb{Q}(\zeta_\ell)$. The field $\mathcal{F}_\ell$ is Galois over $\mathcal{F}_1 = \mathbb{Q}(j)$ with Galois subgroup $\mathrm{Gal}(\mathcal{F}_\ell/\mathcal{F}_1(\zeta_\ell)) \simeq \Gamma(1)/\Gamma(\ell)\cdot\{\pm 1\}$. In [11], the author gives a pair of generators X_2 and X_3 for $\mathcal{F}_\ell$, for prime $\ell \geq 5$, (see also [9, 10] for the more general setting) with $q-$product expansions

$$X_2 = q^{-(k-1)/\ell}\prod_{n\geq 1}\frac{(1-q^{\ell n-2})(1-q^{\ell n-(\ell-2)})}{(1-q^{\ell n-1})(1-q^{\ell n-(\ell-1)})} \tag{5.17}$$

$$X_3 = q^{-(2k-3)/\ell}\prod_{n\geq 1}\frac{(1-q^{\ell n-3})(1-q^{\ell n-(\ell-3)})}{(1-q^{\ell n-1})(1-q^{\ell n-(\ell-1)})}. \tag{5.18}$$

By (4.10) and (4.11) we have $X_2 = \sigma_k^{-1}((-1)^{k-1}r_{\ell,1}^{-1}),\ X_3 = \sigma_k^{-1}(-r_{\ell,1}^{-1}r_{\ell,2})$, $\ell > 5$. Hence

$$\begin{aligned}\mathcal{F}_\ell \cong \sigma_k(\mathcal{F}_\ell) &= \sigma_k(\mathbb{Q}(\zeta_\ell, X_2, X_3)\\ &= \mathbb{Q}(\zeta_\ell, \sigma_k(X_2), \sigma_k(X_3))\\ &= \mathbb{Q}(\zeta_\ell, r_{\ell,1}^{-1}, r_{\ell,1}^{-1}r_{\ell,2})\\ &= \mathbb{Q}(\zeta_\ell, r_{\ell,1}, r_{\ell,2}). \end{aligned} \tag{5.19}$$

However, the functions $r_{\ell,j}$ are modular units by Theorem 1.1 *i.*, so that $\mathbb{Q}(\zeta_\ell, r_{\ell,1}, r_{\ell,2}) \subseteq \mathcal{F}_\ell$. The automorphism σ_k is of finite order, so that $\sigma_k(\mathcal{F}_\ell) \subseteq \mathcal{F}_\ell$ implies $\sigma_k(\mathcal{F}_\ell) = \mathcal{F}_\ell$, and thus for $\ell > 5$,

$$\mathbb{Q}(\zeta_\ell, r_{\ell,1}, r_{\ell,2}) = \mathcal{F}_\ell.$$

For $\ell = 5$, the genus of the associated modular curve $X(5) \simeq (\Gamma(5)\backslash\mathcal{H})^*$ is zero, thus there exists a hauptmodul, a modular function in $\mathcal{F}_5$ with exactly one simple pole. It is well known, and not difficult to see, that $r_{5,1}(\tau)$ satisfies this property, which concludes the proof of Theorem 1.1 *ii*.

6. Gauss periods

6.1. Gauss periods. Given an odd prime ℓ with factorization $\ell = fk + 1$, $f \geq 1$, there is a unique subfield K of $\mathbb{Q}(\zeta_\ell)$ with $[\mathbb{Q}(\zeta_\ell) : K] = f$. If we let $\{m_n\}$, $1 \leq n \leq k$ be a set of representatives for the cosets of $(\mathbb{Z}/\ell\mathbb{Z})^\times/\big((\mathbb{Z}/\ell\mathbb{Z})^\times\big)^k$, the Gauss periods of degree f are defined for each n by

$$\mathrm{Tr}_{\mathbb{Q}(\zeta_\ell)/K}(\zeta_\ell^{m_n}), \tag{6.1}$$

and have a common minimal polynomial $F_k(x)$ of degree k. In the case $f = 2$ we note the unique subfield of degree 2 is $\mathbb{Q}(\zeta_\ell)^+$, and we may choose $m_n = n$. In this case Gauss explicitly described the coefficients d_{k-j} of x^j, $0 \leq j \leq k$, of the degree k minimal polynomial $F_k(x)$ by

$$d_{k-j} = (-1)^{\lfloor \frac{k-j}{2} \rfloor} \binom{\lfloor \frac{k+j}{2} \rfloor}{\lfloor \frac{k-j}{2} \rfloor}. \tag{6.2}$$

Later Sylvester indicated how one may obtain the coefficients for composite ℓ and $f = 2$ recursively, and gave a list of the polynomials for $1 \leq \ell \leq 36$. The results of Gauss and Sylvester may be found in [28]. The period polynomials for various ℓ, f and k have since been investigated. In [8], Gupta and Zagier consider the case $f = 2$, and extend the definition (6.1) to odd ℓ. In [8] the authors prove the reciprocal polynomial

$$f_k(x) = x^k F_k(x^{-1}) \tag{6.3}$$

with roots $\{1/\mathrm{Tr}_{\mathbb{Q}(\zeta_\ell)/K}(\zeta_\ell^n) \mid 1 \leq n \leq k\}$ has coefficients d_{k-j} of x^{k-j} in agreement with those given by Gauss in (6.2). Here, we recover this original result of Gauss, and the more general result of Gupta and Zagier by different means. In particular, we observe this result after considering the limiting value of a functional equation satisfied by the modular unit $r_{\ell,1}$. To prove Theorem 1.6, we will use the following lemmas. The first follows from the fact that

$$\lim_{\tau \to 0} \frac{1 - q^a}{1 - q} = a.$$

Lemma 6.1. *For integers $r, j \geq 0$ and $k \geq 1$,*

$$\lim_{\tau \to 0} s_{k,2r}(q^j) = (-1)^r \sum_{n=0}^{m_1 - r - 1} \binom{n+r}{n} \binom{m_1 - (n+1)}{m_1 - (n+r+1)}$$

$$\lim_{\tau \to 0} s_{k,2r-1}(q^j) = (-1)^{r+1} \sum_{n=0}^{m_2 - r} \binom{n+r}{n} \binom{m_2 - (n+1)}{m_2 - (n+r)}$$

where $m_1 = \lfloor \frac{k+1}{2} \rfloor$ and $m_2 = \lfloor \frac{k}{2} \rfloor$.

Lemma 6.2. *For integers $m, n \geq 0$ and $j \geq 1$,*

$$\lim_{\tau \to 0} \quad S_k^{(j)}(q^n) - S_k^{(j)}(q^m) = 0$$

Lemma 6.3. *For integers $m, r \geq 0$ such that the following expressions are defined, we have*

$$\sum_{n=0}^{m-2r-1} \binom{r+n}{n} \binom{m-r-n-1}{m-2r-n-1} = \binom{m}{2r+1} \tag{6.4}$$

$$\sum_{n=0}^{m-2r} \binom{r+n}{n} \binom{m-r-n-1}{m-2r-n} = \binom{m}{2r} \tag{6.5}$$

Proof. (Lemma 6.3.)
For $r \geq 0$ we have

$$(1-x)^{-r-1} = \sum_{n=0}^{\infty} \binom{r+n}{n} x^n \tag{6.6}$$

so that x^n has coefficient $\binom{r+n}{n}$ and $x^{m-2r-n-1}$ has coefficient $\binom{m-r-n-1}{m-2r-n-1}$, where $0 \leq n \leq m-2r-1$. We find that x^{m-2r-1} in the product $(1-x^{-2r-2})$ has coefficient given by the left hand side of (6.4), but also $\binom{m}{m-2r-1} = \binom{m}{2r+1}$ by (6.6). This gives (6.4), and we argue similarly to establish (6.5). □

Proof. (Lemma 6.2.)
For $j = 1$ we have for any $r \geq 0$

$$\lim_{\tau \to 0} S_k{}^{(1)}(q^r) = \lim_{\tau \to 0} s_{k,0}(q^r) = \lim_{\tau \to 0} \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor - 1} q^{i(r+1)} = \left\lfloor \frac{k+1}{2} \right\rfloor.$$

By induction, the result follows for $j > 1$ from Lemma 6.1 and the fact that

$$S_k^{(j)}(q^n) = \sum_{i=0}^{k-1} s_{k,i}(q^n) S_k^{(j-(i+1))}(q^{n+i+1}).$$

□

Proof. (Theorem 1.6.)
That the modular function $r_{\ell,1}(\tau)$ satisfies the equation $M_k(X)$ defined in Theorem 1.6 follows from (3.4), (3.7) and (3.15). If $\lim_{\tau \to 0} M_k(X) = m_k(x)$

has coefficients c_{k-j} of x^{k-j}, by (1.7) and the Lemmas, we have for j even

$$\begin{aligned}
c_{k-j} &= (-1)^{j/2} \sum_{n=0}^{\lfloor\frac{k}{2}\rfloor-\frac{j}{2}} \binom{\frac{j}{2}+n}{n}\binom{\lfloor\frac{k}{2}\rfloor-(n+1)}{\lfloor\frac{k}{2}\rfloor-\frac{j}{2}-n} \\
&= (-1)^{j/2} \sum_{n=0}^{\lfloor\frac{k}{2}\rfloor-\frac{j}{2}} \binom{\frac{j}{2}+n}{n}\binom{\lfloor\frac{k-j}{2}\rfloor+\frac{j}{2}-n-1}{\lfloor\frac{k}{2}\rfloor-\frac{j}{2}-n} \\
&= (-1)^{j/2}\binom{k-\lfloor\frac{k-j+1}{2}\rfloor}{j} \\
&= (-1)^{j/2}\binom{k-\lfloor\frac{k-j+1}{2}\rfloor}{\lfloor\frac{k-j}{2}\rfloor} \\
&= (-1)^{j/2}\binom{\lfloor\frac{k+j}{2}\rfloor}{\lfloor\frac{k-j}{2}\rfloor} \qquad (6.7)
\end{aligned}$$

and for j odd

$$\begin{aligned}
c_{k-j} &= (-1)^{(j+1)/2} \sum_{n=0}^{\lfloor\frac{k+1}{2}\rfloor-\frac{j-1}{2}-1} \binom{\frac{j-1}{2}+n}{n}\binom{\lfloor\frac{k+1}{2}\rfloor-(n+1)}{\lfloor\frac{k+1}{2}\rfloor-\frac{j-1}{2}-n-1} \\
&= (-1)^{(j+1)/2} \sum_{n=0}^{\lfloor\frac{k+1}{2}\rfloor-\frac{j-1}{2}-1} \binom{\frac{j-1}{2}+n}{n}\binom{\lfloor\frac{k-j}{2}\rfloor+\frac{j-1}{2}-n-1}{\lfloor\frac{k}{2}\rfloor-\frac{j-1}{2}-n} \\
&= (-1)^{(j+1)/2}\binom{k-\left\lfloor\frac{k-j+1}{2}\right\rfloor}{j} \\
&= (-1)^{(j+1)/2}\binom{k-\left\lfloor\frac{k-j+1}{2}\right\rfloor}{\lfloor\frac{k-j}{2}\rfloor} \\
&= (-1)^{(j+1)/2}\binom{\lfloor\frac{k+j}{2}\rfloor}{\lfloor\frac{k-j}{2}\rfloor}. \qquad (6.8)
\end{aligned}$$

We use the identity $\binom{m}{m-r}=\binom{m}{r}$, the fact that $k-\lfloor\frac{k-j+1}{2}\rfloor-\lfloor\frac{k-j}{2}\rfloor=j$, and that

$$\left\lfloor\frac{k-j}{2}\right\rfloor = \begin{cases} \lfloor\frac{k}{2}\rfloor-\frac{j}{2} & j \text{ even} \\ \lfloor\frac{k+1}{2}\rfloor-\frac{j-1}{2}-1 & j \text{ odd} \end{cases}$$

and conclude that the limiting value $\lim_{\tau\to 0} r_{\ell,1}(\tau)$ satisfies the polynomial $m_k(x)$, with coefficients given by (6.7) and (6.8), and that this polynomial is integral. □

Proof. (Corollary 1.1) In [8], the authors prove via polynomial recurrences that for the polynomial $f_k(x)$ as defined by (6.3), the coefficients d_{k-j} of x^{k-j} are given by (6.2). Comparing (6.7) and (6.8), we find

$$d_{k-j} = \begin{cases} (-1)^{\lfloor \frac{k+1}{2} \rfloor} c_{k-j} & j \text{ odd} \\ (-1)^{\lfloor \frac{k}{2} \rfloor} c_{k-j} & j \text{ even} \end{cases}$$

and conclude

$$m_k(x) = (-1)^{\lfloor \frac{k+1}{2} \rfloor} f_k((-1)^k x). \tag{6.9}$$

By (4.15) we deduce

$$\lim_{\tau \to 0} r_{\ell,1}(\tau) = \frac{(-1)^k}{\mathbf{Tr}_{F/F^+}\left(\zeta_\ell^{\lfloor \frac{k+1}{2} \rfloor}\right)} \tag{6.10}$$

where $F = \mathbb{Q}(\zeta_\ell)$. Thus the roots of $m_k(x)$ are given by

$$\{(-1)^k / \mathrm{Tr}_{\mathbb{Q}(\zeta_\ell)/K}(\zeta_\ell^n) \mid 1 \leq n \leq k\}, \tag{6.11}$$

which is equal to the union over the divisors $d > 1$ of ℓ of the Galois conjugates of the cyclotomic units $\lim_{\tau \to 0}(-1)^{k-1} r_{d,1}(\tau)$, proving Corollary 1.1. □

7. Class field theory

We next examine results within class field theory related to the modular units $r_{\ell,j}$. We note that Theorem 1.3 requires only one specialization of one modular function to generate the field K_ℓ and compare the following classical result.

Theorem R. *For $K = \mathbb{Q}(z)$ an imaginary quadratic field, and any positive integer N, the ray class field of conductor N over K is given by the field $K \cdot \mathcal{F}_N|_z$, where*

$$\mathcal{F}_N|_z = \{f(z) \mid f \in \mathcal{F}_N, f(z) \neq \infty\}.$$

That is $K_N = K \cdot \mathcal{F}_N|_z$.

Proof. (Theorem R.) See [27], or [20]. □

Example. To illustrate Theorem 1.3, with $\tau = i$, the ray class field of conductor 5 over $K = \mathbb{Q}(i)$ is given by

$$\mathbb{Q}(i)_5 = \mathbb{Q}(i, r(i))$$

where

$$r(\tau) = \frac{q^{1/5}}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1 + \cdots}$$

is the Rogers-Ramanujan continued fraction. The special value $r(i)$, also determined by Ramanujan, is given by

$$r(i) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}.$$

As another example we may take $z = \rho$, and find

$$\mathbb{Q}(\rho)_5 = \mathbb{Q}(\rho, r(\rho)),$$

where

$$r(\rho) = e(-1/10)\frac{\sqrt{30+6\sqrt{5}} - 3 - \sqrt{5}}{4},$$

also given by Ramanujan.

7.1. Shimura reciprocity. To prove Theorems 1.3 and 1.4, we will use in part the language of varieties as in [27]. Let V and W be rational varieties defined over a common field k. A subvariety $T \subseteq V \times W$ is called a rational map if $(v, w) \in T \Rightarrow k(v, w) = k(v)$. A rational map is called a rational function if $W = \mathbb{A}^1$ one dimensional affine space. The rational functions on V form a field, which we denote by $k(V)$.

Let

$$\begin{aligned} G_{(p)} &= \mathrm{GL}_2(\mathbb{Q}_p) \\ G_\infty &= \mathrm{GL}_2(\mathbb{R}) \\ G_A &= \{(x) \in \prod_p G_p \cdot G_\infty \mid x_p \in \mathrm{GL}_2(\mathbb{Z}_p) \text{ for almost all } p.\} \\ G_0 &= \{(x) \in G_A \mid x_\infty = 1\} \\ G_A^+ &= G_0 G_\infty^+ \\ \mathcal{Z} &= \{S \subseteq G_A^+ \mid S \text{ open}, \ \mathbb{Q}^* G_\infty^+ \subseteq S, \ S/\mathbb{Q}^* G_\infty^+ \text{ compact.}\} \end{aligned}$$

We note that a topology is placed on G_A by declaring the set G, as defined in (3.12), to be open. Let $S \subseteq \mathrm{G}_{A+}$ be an open subset of G_{A+} such that $\mathbb{Q}^*\mathrm{G}_{\infty+} \subseteq S$ and $S/\mathbb{Q}^*\mathrm{G}_{\infty+}$ is compact, and let $k_S \subset \mathbb{Q}^{\mathrm{ab}}$ be the Abelian extension of $\mathbb{Q}$ fixed by $\mathbb{Q}^*\det(S)$. Then there is a natural variety V_S corresponding to S defined over k_S as follows. Given any Fuchsian group Γ of the first kind, $\Gamma\backslash\mathcal{H}^*$ is a compact Riemann surface, so there exists a nonsingular algebraic curve V defined over a subfield of $\mathbb{C}$ so that $\Gamma\backslash\mathcal{H}^*$ is biregularly isomorphic to V. We say (V, ϕ) is a model of $\Gamma\backslash\mathcal{H}^*$ if ϕ is a Γ-invariant holomorphic map, $\phi : \mathcal{H}^* \longrightarrow V$, that yields such an isomorphism. If we let

$$\mathcal{F}_S = \{h \in \mathcal{F} \mid h^{\sigma(s)} = h \ \forall\ s \in S\} \tag{7.1}$$

$$\Gamma_S = S \cap \mathrm{G}_{\mathbb{Q}+} \tag{7.2}$$

then there exists a model (V_S, ϕ_S) of $\Gamma_S\backslash\mathcal{H}^*$ such that

$$V_S \text{ is defined over } k_S \tag{7.3}$$

$$\mathcal{F}_S = \{f \circ \phi_S \mid f \in k_S(V_S)\}. \tag{7.4}$$

We summarize some known results in the following proposition.

Proposition 7.1. *Each set $S \in \mathcal{Z}$ defines a corresponding subfield $k_S \subseteq \mathbb{Q}^{\mathrm{ab}}$ of finite index.*

Proof. (Proposition 7.1.) The composition

$$G_A \xrightarrow{\det} \mathbb{Q}_A^* \xrightarrow{[\ (-)^{-1},\ \mathbb{Q}\ ]} \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$$

defines a homomorphism $\varphi : G_A \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$. We note

$$\mathbb{Q}^* \cdot \det(S) \subseteq \mathbb{Q}_A^*,$$

and take the field k_S so that

$$\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/k_S) = \varphi(S).$$

□

Now for $\tau \in \mathbb{C} - \mathbb{R}$ imaginary quadratic, there is an embedding E_τ (see [27] 4.4)

$$E_\tau : K \hookrightarrow \mathrm{M}_2(\mathbb{Q})$$

such that

$$E_\tau(K^*) = \{\gamma \in \mathrm{GL}_2(\mathbb{Q})^+ \mid \gamma(\tau) = \tau\},$$

and for $k \in K^*$,

$$E_\tau(k)\begin{bmatrix} \tau \\ 1 \end{bmatrix} = k\begin{bmatrix} \tau \\ 1 \end{bmatrix} \tag{7.5}$$

The embedding extends naturally to K_A^*, the idèle group of K. We will use the following proposition, with the correspondence given in the sense of Proposition 7.1.

Proposition S. *(Shimura [27]) Let $S \in \mathcal{Z}$, $\tau \in \mathbb{C} - \mathbb{R}$ imaginary quadratic, and let*

$$W = \{s \in K_A^* \mid E_\tau(s) \in S\}.$$

Then $K \cdot k_S(\phi_S(\tau)) \subseteq K^{\mathrm{ab}}$ corresponds to the subgroup

$$K^*W \subseteq K_A^*.$$

Example. As an illustration of the proposition, for any $N \in \mathbb{N}$, consider the set

$$S_N = \mathbb{Q}^*\{(x) \in G \mid x_p \equiv 1 \mod N \cdot \mathrm{M}_2(\mathbb{Z}_p)\}.$$

Then $\Gamma_{S_N} = \mathbb{Q}^*\Gamma(N)$, so that (V_{S_N}, ϕ_{S_N}) is a model of $\Gamma(N)\backslash\mathcal{H}^*$ over $\mathbb{Q}$. We will use the following known result.

Proposition. *With $S = S_N$ as above, the corresponding set W is given by*

$$\mathbb{Q}^* \backslash W = W_N = \prod_p (1 + N\mathcal{O}_p)^* \subset K_A^*. \tag{7.6}$$

In what follows, we let τ be the variable on the upper half plane $\mathcal{H}$. We let ϕ_N be the biregular homomorphism associated to $\Gamma_{S_N} \backslash \mathcal{H}^*$, and let V_N be the locus of

$$\phi_N(\tau) = (j(\tau), f^1_{a_1}(\tau), f^2_{a_1}(\tau), f^3_{a_1}(\tau), f^1_{a_2}(\tau), f^2_{a_2}(\tau), f^3_{a_2}(\tau), \dots)$$

where $\{a_i\}_i = \frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2 - \{0\}$, and for the lattice $L = L(\tau, 1)$, the Fricke functions f^i_a are defined by

$$f^1_a(\tau, L) = \frac{g_2(L) g_3(L)}{\Delta(L)} \wp\Big(a\Big[\begin{smallmatrix}\tau\\1\end{smallmatrix}\Big]; L\Big)$$

$$f^2_a(\tau, L) = \frac{g_2(L)^2}{\Delta(L)} \wp\Big(a\Big[\begin{smallmatrix}\tau\\1\end{smallmatrix}\Big]; L\Big)^2$$

$$f^3_a(\tau, L) = \frac{g_3(L)}{\Delta(L)} \wp\Big(a\Big[\begin{smallmatrix}\tau\\1\end{smallmatrix}\Big]; L\Big)^3.$$

where $\wp$ is the Weierstrass $\wp-$function, and g_2, g_3 invariants of the lattice (see [20]).

Two curves are birational if and only if their function fields are isomorphic, and there exists a birational map $X : V_{S_N} \longrightarrow V_N$ such that $X \circ \phi_{S_N} = \phi_N$. Thus the map X gives an isomorphism

$$k_{S_N}(V_{S_N}) \xrightarrow{X} k_N(V_N)$$

hence

$$K \cdot k_{S_N}(\phi_{S_N}(z)) \cong K \cdot k_N(\phi_N(z)).$$

It is known that the field of modular functions of level N is generated by the modular invariant j and the first Fricke functions f^1_a, where $a \in \frac{1}{N}\mathbb{Z}^2$, $a \notin \mathbb{Z}^2$. Thus, the field $K \cdot \mathcal{F}_N|_z \subseteq K^{\mathrm{ab}}$ corresponds to K^*W. Combining the above results, we arrive at the following known result.

$$\mathrm{Gal}(K^{\mathrm{ab}}/K_N) \cong \prod_p (1 + N\mathcal{O}_p)^*. \tag{7.7}$$

7.2. Galois action on U^C_N.

Proposition 7.2. *i. For fixed m, $1 \le m \le k-1$, one has*

$$\begin{aligned}\mathrm{Gal}(\mathcal{F}/\mathcal{F}_1(r_{\ell,m})) &\cong \Big\{\sigma = (\sigma_p) \in G \mid \sigma_p \equiv \left(\begin{smallmatrix}1&0\\0&s\end{smallmatrix}\right) \bmod \ell \cdot \mathrm{M}_2(\mathbb{Z}_p)\Big\}\Big/ G^+_\infty G_\ell \\ &\cong (\mathbb{Z}/\ell\mathbb{Z})^*_,\end{aligned}$$

where $s \in \mathbb{Z}$ *is such that* $s \not\equiv 0 \bmod \ell$.

ii. In particular, for fixed m, $2 \le m \le k$, *if*

$\gamma \equiv \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \bmod \ell M_2(\mathbb{Z}_\ell)$, *with* $d \not\equiv 0 \mod \ell$, *then one has*

$$(r_{\ell,k+1-m})^\gamma = e(-(kb(m^2-1)/\ell))r_{\ell,k+1-m}.$$

Proof. (Theorem 1.3.) Let

$$S_r := \left\{\sigma = (\sigma_p) \in G \mid \sigma_p \equiv \left(\begin{smallmatrix} 1 & n_s \\ 0 & s \end{smallmatrix}\right) \bmod \ell \cdot \mathrm{M}_2(\mathbb{Z})\right\}$$

where $s \not\equiv 0 \mod \ell$. Assuming Proposition 7.2, as in Proposition S take $S = S_r$ and suppose $u \in W$. For any m, $1 < m \le (\ell-1)/2$, $u_p \equiv s \mod \ell\mathbb{Z}_p$. By the property given in (7.5) defining the embedding E_τ, we find $s \equiv 1 \bmod \ell\mathbb{Z}_p$, so that $u_p \in 1 + \ell\mathcal{O}_p$, hence $u \in \prod_p (1 + \ell\mathcal{O}_{K_p})^*$. To prove the converse, $u \in 1 + \ell\mathcal{O}_K$ implies $E_\tau(u) \equiv 1 \bmod \ell$, so that $E_\tau(u) \in S$, hence $u \in W$. Theorem 1.3 follows from the following proposition. □

Proposition 7.3. *For* S *and* τ *as in Proposition S, and* $\mathcal{F}_S$ *as in (7.1), if there exists some* $h \in \mathcal{F}_S$ *such that* h *is non-singular at* τ, *and* $\mathcal{F}_S = \mathcal{F}_1(h)$, *then*

$$K \cdot k_S(\phi_S(\tau)) = K \cdot k_S(j(\tau), h(\tau)).$$

Proof. (Proposition 7.3.) We have $h \in \mathcal{F}_S$, so by (7.4) there exists some $f \in k_S(V_S)$ such that $h = f \circ \phi_S$, so that $h(\tau) = f \circ \phi_S(\tau)$, which is non-singular by hypothesis. $f \in k_S(V_S)$ implies $k_S(\phi_S(\tau), f(\phi_S(\tau)) = k_S(\phi_S(\tau))$ so that $h(\tau) \in K \cdot k_S(\phi_S(\tau))$. The same argument holds replacing h by j. Thus, $K \cdot k_S(j(\tau), h(\tau)) \subseteq K \cdot k_S(\phi_S(\tau))$. To prove the reverse inclusion, without loss of generality, we may write $\phi_S(\tau) = (\tau_1, \tau_2, \tau_3, \dots)$. Let P_j denote the projection onto the j-th coordinate, that is, define $P_j : V_S \to \mathbb{C} \simeq \mathbb{A}^1$ by $P_j(v) = \tau_j$, where $v = (\tau_1, \tau_2, \tau_3, \dots)$, and $j \ge 1$. Then $k_S(P_j(v), v) = k_S(\tau_j, \tau_1, \tau_2, \tau_3, \dots) = k_S(\tau_1, \tau_2, \tau_3, \dots) = k_S(v)$. Thus P_j is a rational map of V_S. This implies $P_j \circ \phi_S \in \mathcal{F}_S = \mathcal{F}_1(h)$, where the last equality holds by hypothesis. Thus, there exist $f_i \in \mathcal{F}_1$, $1 \le i \le N$, for some $N \in \mathbb{N}$, such that $P_j \circ \phi_S = \sum_{i=1}^N f_i \cdot h^i$. This implies $P_j(\tau) = \tau_j = \sum_{i=1}^N f_i(\tau) \cdot h^i(\tau)$. But $\mathcal{F}_1 = \mathbb{Q}(j)$ so $f_i(\tau) \in \mathbb{Q}(j(\tau))$. This is true for all $j \ge 1$, so that $\phi_S(\tau) \in k_S(j(\tau), \phi_S(\tau))$. This shows $K \cdot k_s(j(\tau), h(\tau)) = K \cdot k_s(\phi_S(\tau))$. □

Proof. (Proposition 7.2.) By Proposition 5.1, the Galois action on the Siegel functions (5.9), (3.20) and (4.8), we find

$$r^{\gamma}_{\ell,\frac{\ell+1}{2}-m} = \left((-1)^{m-1} e\left(-\frac{k(m-1)}{2\ell}\right) \prod_{s=0}^{\ell-1} \frac{g_{(m/\ell,\ s/\ell)}}{g_{(1/\ell,\ s/\ell)}}\right)^{\gamma}$$
$$= (-1)^{m-1} e\left(-\frac{k(m-1)}{2\ell}\right) \prod_{s=0}^{\ell-1} \frac{g_{(m/\ell,\ s/\ell)\gamma}}{g_{(1/\ell,\ s/\ell)\gamma}}$$
$$= (-1)^{m-1} e\left(-\frac{k(m-1)}{2\ell}\right) \prod_{s=0}^{\ell-1} \frac{g_{((am+cs)/\ell,\ (bm+ds)/\ell)}}{g_{((a+cs)/\ell,\ (b+ds)/\ell)}}. \tag{7.8}$$

Let us assume $\gamma \in \mathrm{Gal}(\mathcal{F}/\mathcal{F}_1(r_{\ell,\frac{\ell+1}{2}-m})$. We first show that $c \equiv 0 \bmod \ell$. If we suppose $c \not\equiv 0 \bmod \ell$, then both of the sets $\{am+cs \mid 0 \le s \le \ell-1\}$ and $\{a+cs \mid 0 \le s \le \ell-1\}$ contain a complete set of representatives for $\mathbb{Z}/\ell\mathbb{Z}$. We choose $t_s \equiv am+cs \mod \ell$, $r_s \equiv a+cs \mod \ell$, where $0 \le t_s, r_s \le \ell-1$, and find by (5.4)

$$g_{((am+cs)/\ell,*)} = \kappa_{s_m} g_{(t_s,*)}$$
$$g_{((a+cs)/\ell,*)} = \kappa_{s_1} g_{(r_s,*)}$$

$\kappa_{s_m}, \kappa_{s_1} \in \mathbb{C}$. Thus, using (5.8), we find

$$\mathrm{ord}_\infty r^{\gamma}_{\ell,\frac{\ell+1}{2}-m} = \frac{\ell}{2}\sum_{s=0}^{\ell-1} \mathbf{B}_2(t_s) - \mathbf{B}_2(r_s) = 0.$$

However we have

$$\mathrm{ord}_\infty r_{\ell,\frac{\ell+1}{2}-m} = \frac{1}{2}((m^2-1) - \ell(m-1))$$
$$= \frac{1}{2}(m-1)(m-\ell+1) \neq 0$$

as $1 < m \le \frac{\ell-1}{2}$, and thus $c \equiv 0 \bmod \ell$.

We now argue that $a \equiv \pm 1 \bmod \ell$. We may write $(am+cs)/\ell = am/\ell + b_s$, $(a+cs) = a/\ell + b_s$, where $b_s \in \mathbb{Z}$. Thus,

$$r^{\gamma}_{\ell,\frac{\ell+1}{2}-m} = (-1)^{m-1} e\left(-\frac{k(m-1)}{2\ell}\right) \kappa \prod_{s=0}^{\ell-1} \frac{g_{(am/\ell,*)}}{g_{(a/\ell,*)}} \tag{7.9}$$

where κ is a constant determined by (5.4) and (5.5). By (7.9), we have

$$\mathrm{ord}_\infty r^{\gamma}_{\ell,\frac{\ell+1}{2}-m} = \sum_{s=0}^{\ell-1} \frac{\ell}{2}(\mathbf{B}_2(am/\ell) - \mathbf{B}_2(a/\ell)) = \frac{1}{2}(a^2(m^2-1) - \ell a(m-1)).$$

Then $\mathrm{ord}_\infty r_{\ell,\frac{\ell+1}{2}-m} \equiv \mathrm{ord}_\infty r^{\gamma}_{\ell,\frac{\ell+1}{2}-m} \mod \ell \Leftrightarrow a^2(m^2-1) \equiv (m^2-1) \mod \ell \Leftrightarrow a \equiv \pm 1 \mod \ell$, where we use the fact that ℓ is prime, $m \not\equiv \pm 1$

mod ℓ. As γ is identified with $-\gamma$ under the Galois action, we have $a \equiv 1$ mod ℓ.

Let

$$\mu' = ((m+1)b + (d-1)(\ell-1)/2 - (a-1))/\ell.$$

We may assume m is odd, for otherwise, we replace m by $\ell - m$ which is odd, and note that $\prod_{s=0}^{\ell-1} g_{(m/\ell,\ s/\ell)} = \prod_{s=0}^{\ell-1} g_{((\ell-m)/\ell,\ s/\ell)}$. We point out that $(m^2-1)b \equiv 0 \mod \ell \Leftrightarrow \left(\frac{(d-1)k}{\ell} - \mu'\right) k(m-1) \in \mathbb{Z}$, as $a \equiv 1 \mod \ell$. Next we show that

$$(r_{\ell,k+1-m})^{\gamma} = e\left(-\frac{kb}{\ell}(m^2-1)\right) r_{\ell,k+1-m}, \tag{7.10}$$

where $\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{Gal}\left(\mathcal{F}_\ell/\mathcal{F}_1\right), d \not\equiv 0 \mod \ell$. By (5.8), one has

$$g_{\left(\frac{a_1}{\ell}, 1+\frac{a_2}{\ell}\right)} = e\left(\frac{a_1-\ell}{2\ell}\right) g_{\left(\frac{a_1}{\ell}, \frac{a_2}{\ell}\right)} = (-1)^{a_1-1} e\left(-\frac{ka_1}{\ell}\right) g_{\left(\frac{a_1}{\ell}, \frac{a_2}{\ell}\right)}.$$

Applying this bm times with $a_1 = m$, one has

$$\prod_{s=0}^{\ell-1} g_{(m/\ell,(bm+s)/\ell)} = e\left(-\frac{kbm^2}{\ell}\right) \prod_{s=0}^{\ell-1} g_{(m/\ell,s/\ell)}, \tag{7.11}$$

for $m(m-1) \in 2\mathbb{Z}$. Similarly, one finds

$$\prod_{s=0}^{\ell-1} g_{(1/\ell,(b+s)/\ell)} = e\left(-\frac{kb}{\ell}\right) \prod_{s=0}^{\ell-1} g_{(1/\ell,s/\ell)}. \tag{7.12}$$

Recalling (7.8), (7.10) now follows by taking the quotient of (7.11) and (7.12). □

Proof. (Theorem 1.4.) To prove Theorem 1.4, for fixed $\ell \in \mathbb{Z}^+$, and any $m \in \mathbb{Z}^+$, any $b_i, n_i \in \mathbb{Z}$, we define the product

$$h(L) = \prod_{i=1}^{m} g_{x_i}\left(\frac{\sigma_{b_i} L}{\mathbf{N}(\sigma_{b_i})}\right)^{12\ell n_i} \tag{7.13}$$

where $\mathbf{N}(\sigma) = \det(\sigma)$, $\sigma_{b_i} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & b_i \end{smallmatrix}\right)$, and where $x_i \in \frac{1}{\ell}\mathbb{Z}^2 \setminus \mathbb{Z}^2$. We now show that $h(L)$ is modular with respect to a particular subgroup of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Homogeneity and the existence of a q-expansion follow from known properties of the Klein forms $\mathfrak{t}_a$ (5.2), (5.3), (5.4) and by definition (5.1).

In what follows, to ease notation we may write $\prod_{i=1}^m = \prod$. If we take $\gamma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma(\ell \prod b_i)$, we have

$$\begin{aligned} h(\gamma L) &= \prod g_{x_i}\left(\frac{\sigma_{b_i}\gamma L}{\mathbf{N}(\sigma_{b_i})}\right)^{12\ell n_i} = prod g_{x_i}\left(\gamma_i \frac{\sigma_{b_i} L}{\mathbf{N}(\sigma_{b_i})}\right)^{12\ell n_i} \\ &= \prod g_{x_i\gamma_i}\left(\frac{\sigma_{b_i} L}{\mathbf{N}(\sigma_{b_i})}\right)^{12\ell n_i} = \prod (\epsilon(x_i,\gamma_i) g_{x_i}\left(\frac{\sigma_{b_i} L}{\mathbf{N}(\sigma_{b_i})}\right)^{12\ell n_i} \\ &= \prod \epsilon_{x_i}(\gamma_i)^{12\ell n_i} \cdot h(L), \end{aligned}$$

where $\gamma_i = \left(\begin{smallmatrix} \alpha & \beta b_i^{-1} \\ \gamma b_i & \delta \end{smallmatrix}\right) \in \Gamma(\ell)$, and $\epsilon_{x_i}(\gamma_i)$ is a 2ℓ-th root of unity (see [19], Chapter 2). We find that the product $\prod \epsilon_{x_i}(\gamma_i)^{12\ell n_i} = 1$, so that h is invariant under the action of $\Gamma(\ell \prod b_i)$. By examining the q-expansion of $h(L)$, we verify that the coefficients in fact lie in $\mathbb{Q}(\zeta_\ell)$, so that $h(L)$ is modular with respect to $\Gamma(\ell \prod b_i)$. We now describe the Galois action on $h(L)$, and let $\mathcal{G} = \mathrm{GL}_2(\mathbb{Z}/\ell \prod b_i\mathbb{Z})/\pm 1$.

Proposition 7.4. *Let $\sigma \in \mathcal{G}$. Then the Galois action of $\mathcal{G}$ on h is given by*

$$h(L)^\sigma = \prod_{i=1}^m g_{x_i\gamma_i'\sigma_\delta}\left(\frac{\sigma_{b_i}' L}{\mathbf{N}(\sigma_{b_i}')}\right)^{12\ell n_i} \tag{7.14}$$

where $\sigma = \gamma\sigma_\delta, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/\ell \prod_i b_i\mathbb{Z})$, $\sigma_\delta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta \end{smallmatrix}\right)$, $\delta \in (\mathbb{Z}/\ell \prod_i b_i\mathbb{Z})^*$, and $\gamma_i' \in \mathrm{SL}_2(\mathbb{Z}/\ell \prod_i b_i\mathbb{Z})$, σ_{b_i}' are defined by $\sigma_{b_i}' = \gamma_i'^{-1}\sigma_{b_i}\gamma = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$.

Proof. (Proposition 7.4.) As discussed in Section 1.1, $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is generated by $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ and the matrices $\gamma_d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d \end{smallmatrix}\right)$, $d \in (\mathbb{Z}/N\mathbb{Z})^*$. We now show that a decomposition

$$\gamma_i'^{-1}\sigma_{b_i}\gamma = \sigma_{b_i}' = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \tag{7.15}$$

exists. If $\gamma \in \mathrm{SL}_2(\mathbb{Z}/\ell \prod b_i\mathbb{Z})$, then a and c are relatively prime mod $\ell \prod b_i$. We let $g_i = \text{g.c.d.}(a, b_i) \bmod \ell \prod b_i$, and decompose $a = a_i g_i, b_i = b_i' g_i$. Then there exist X_i and $Y_i = y_i s_1$ such that

$$Y_i a_i + X_i b_i' \equiv 1 \bmod \ell \prod b_i \tag{7.16}$$

where we use the fact that $(b_i, \ell) = 1$. If we let $M_i = g_i d - Y_i bc$, and $N_i = -X_i b$, then the matrix

$$\gamma_i'^{-1} = \left(\begin{smallmatrix} M_i & N_i \\ -cb_i' & a_i \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/\ell \prod b_i\mathbb{Z})$$

gives

$$\sigma'_{b_i} = \begin{pmatrix} g_i & Y_i b \\ 0 & b'_i \end{pmatrix}.$$

Proposition 7.4 follows by applying (7.15), the action of $\mathcal{G}$ on the Siegel functions, and noting that $\mathbf{N}(\sigma_{b_i}) = \mathbf{N}(\sigma'_{b_i})$. $\square$

We note that replacing x by $x+a$ for some $a \in \mathbb{Z}^2$ changes g_x by a root of unity, so that $g_{x+a}^{12\ell} = g_x^{12\ell}$. Thus, we may choose $x = (x_1, x_2)$ with $0 \le \ell x_i < \ell$. Next, we determine $\mathrm{Gal}(\mathcal{F}/\mathcal{F}_1(h))$.

Proposition 7.5. *The Galois group* $\mathrm{Gal}(\mathcal{F}/\mathcal{F}_1(h))$ *is given by*

(7.17)

$$\mathrm{Gal}(\mathcal{F}/\mathcal{F}_1(h)) = \left\{ \sigma = \prod_p \sigma_p = \prod_p \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in G \;\middle|\; \begin{array}{l} a_p \equiv 1 \mod s_1, \\ b_p \equiv 0 \mod \frac{s_1 \prod b_i}{\gcd(a_p, b_i)} \\ c_p \equiv 0 \mod s_2, \\ d_p \equiv 1 \mod s_2 \end{array} \right\}.$$

Assuming Proposition 7.5, we conclude the proof of Theorem 1.4. Let

$$S_h := \left\{ \sigma = \prod_p \sigma_p = \prod_p \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in G \;\middle|\; \begin{array}{l} a_p \equiv 1 \mod s_1, \\ b_p \equiv 0 \mod \frac{s_1 \prod b_i}{\gcd(a_p, b_i)} \\ c_p \equiv 0 \mod s_2, \\ d_p \equiv 1 \mod s_2 \end{array} \right\}$$

denote the set given in (7.17). Using the notation as in Proposition S, let $S = S_h$, and suppose $u \in W$. Then $E_\tau(u) \in S_h$ implies

$$E_\tau(u_p) \begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} a_p\tau + b_p \\ c_p\tau + d_p \end{bmatrix}.$$

The action given in (7.5) also implies $d_p = a_p - c_p\mathbf{T}(\tau)$, $b_p = -c_p\mathbf{N}(\tau)$, where $\mathbf{N}$ and $\mathbf{T}$ denote the norm and trace respectively. Then

$$\begin{aligned} u_p &= c_p(\tau - \mathbf{T}(\tau)) + a_p \\ &= c_p(\tau + m_\tau) - c_p(\mathbf{T}(\tau) + m_\tau) + a_p \end{aligned}$$

Now $a_p = d_p + c_p\mathbf{T}(\tau)$ implies $a_p \equiv 1 \mod s_2$, as $c_p, d_p - 1 \equiv 0 \mod s_2$. Since $a_p \equiv 1 \mod s_1$, we have $a_p \equiv 1 \mod \ell$. We use this, and also the fact that $s_1 \mid \mathbf{T}(\tau) + m_\tau$ to conclude $u_p \in 1 + \mathfrak{f}\mathcal{O}_p$, hence $u \in 1 + \mathfrak{f}\mathcal{O}_K$.

Conversely, let $u \in 1 + \mathfrak{f}\mathcal{O}_K$. If $u = 1 + s_2(\tau + m_\tau)M + \ell N$, then $c \equiv 0 \mod s_2$ and $d = 1 + s_2 m_\tau + \ell N \equiv 1 \mod s_2$. Since $a = c\mathbf{T}(\tau) + d$, we find $a = 1 + s_2 M(\tau + \mathbf{T}(\tau)) + \ell N \equiv 1 \mod s_1$ again using the fact that $s_1 \mid \mathbf{T}(\tau) + m_\tau$. Finally, with hypotheses stated on $\mathbf{N}(\tau)$ and the fact that $b = -c\mathbf{N}(\tau)$, we conclude $q(u) \in S_h$. This shows that $W = 1 + \mathfrak{f}\mathcal{O}_K$.

We now determine k_S. For any $s \in S$, one finds a determinant of the form $\det(s) = 1 + s_1 M_1 + s_2 M_2 + \ell M_3$. Since $(s_1, s_2) = 1$, this determinant may be made arbitrary, so that $k_S = \mathbb{Q}$. We now apply Proposition 7.3, to conclude the proof of Theorem 1.4., □

Proof. (Proposition 7.5.) We will establish necessary and sufficient conditions for the equality of h and h^σ by comparing their divisors. We set $x_i = (r_1 s_2 b_i/\ell, r_2 s_1/\ell)$, $Tx_i = x_i \gamma_i' \sigma_\delta$, where $s_1 s_2 = \ell, r_1, r_2 \in \mathbb{Z}$, and first examine

$$\begin{aligned}
&\mathrm{ord}_\infty h - \mathrm{ord}_\infty h^\sigma \\
&\quad \equiv 12\ell^2 \prod b_i \Big(\sum n_i (\mathbf{B}_2(x_{i_1})/2b_i - g_i^2 \mathbf{B}_2(Tx_{i_1})/2b_i) \Big) \\
&\quad \equiv \sum 6 n_i \ell^2 \prod_{j \neq i} b_j \Big((1 - g_i^2)/6 + x_{i_1}^2 (1 - a^2) - x_{i_1}(1 - g_i a) + x_{i_2} g_i b_i c \\
&\qquad\quad - (x_{i_2} b_i c)^2 - 2 x_{i_1} x_{i_2} a b_i c \Big) \\
&\quad \equiv \sum n_i \prod_{j \neq i} b_j \Big(6 b_i (r_1 s_2)^2 (1 - a^2) - 6 (r_2 s_1 b_i c)^2 - 12 a c b_i^2 r_1 r_2 s_1 s_2 \Big)
\end{aligned}$$

where the congruence is taken mod ℓ. If we assume first that $s_2 \neq 1$, then this difference congruent to 0 implies

$$s_2 \mid 6 (r_2 s_1)^2 \prod b_j c^2 \sum b_i n_i$$

hence $s_2 \mid c^2$. Since ℓ is square free, we must have $s_2 \mid c$. We observe similarly $s_1 \mid a^2 - 1$. We next compare the first Fourier coefficients in the q-expansions of h and h^σ, denoted by $a(h)$ and $a(h^\sigma)$ respectively. We find

$$\begin{aligned}
&a(h)/a(h^\sigma) = \\
&\quad = \prod (e(x_{i_2}(x_{i_1} - 1)/2)/e((Y_i b \mathbf{B}_2(Tx_{i_1})/b_i' + Tx_{i_2}(Tx_{i_1} - 1))/2))^{12\ell n_i} \\
&\quad = e\Big(6\ell \sum \Big(x_{i_2}(x_{i_1} - 1) - Y_i b \mathbf{B}_2(Tx_{i_1})/b_i' - Tx_{i_2}(Tx_{i_1} - 1) \Big) n_i \Big).
\end{aligned}$$

Returning to the definition of x_i and Tx_i, we see that this holds provided

$$\begin{aligned}
6\ell \sum (Y_i b ((a_i' x_{i_1})^2 + (b_i' c x_{i_2})^2 + 1/6)/b_i' \\
+ a_i' \delta b X_i x_{i_1}^2 + b_i' c \delta (g_i d - Y_i b c) x_{i_2}^2) n_i \in \mathbb{Z},
\end{aligned}$$

which holds if and only if

$$\begin{aligned}
&\sum (Y_i b \prod_{j \neq i} b_j (6 a a_i' b_i^2 (r_1 s_2)^2 + 6 b_i b_i' c^2 (r_2 s_1)^2 + g_i \ell^2) n_i \\
&+ 6 \prod_j b_j (a_i' \delta b X_i (r_1 s_2)^2 b_i^2 + b_i' c \delta (g_i d - Y_i b c)(r_2 s_1)^2) n_i \equiv 0 \mod \ell \prod_j b_j.
\end{aligned}$$

This implies

$$s_1 \mid \prod_j b_j ab(r_1 s_2)^2 \sum (Y_i a_i' + X_i b_i' \delta) n_i b_i.$$

We now suppose $s_1 \nmid b$. Then there is some $d_1 \neq 1$, $d_1 \mid s_1$ such that

$$d_1 \mid \delta \sum n_i b_i$$

where we return to the relation given in (7.16), and use the fact that $s_1 \mid Y_i$. Thus, $s_1 \mid b$. We note also that we must have

$$\prod_j b_j \mid \sum Y_i b \prod_{j \neq i} b_j n_i g_i \ell^2$$

a condition that forces $\prod b_i \mid b$.

Finally, we determine conditions under which $\mathrm{ord}_{\mathfrak{a}} h = \mathrm{ord}_{\mathfrak{a}} h^{\sigma}$ for all cusps $\mathfrak{a}$ of $\Gamma(\ell \prod b_i)$. For each cusp $\mathfrak{a}$, there exists some

$$\hat{\gamma}_{\mathfrak{a}} = \left(\begin{smallmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/\ell \prod b_i \mathbb{Z})$$

such that $\gamma_{\mathfrak{a}}^{-1}\infty = \mathfrak{a}$. We first assume $\mathfrak{a} \neq 0$, and as above decompose $\hat{a} = \hat{a_i}\hat{g_i}, b_i = \hat{b_i}\hat{g_i}$, with $\hat{g_i} = \text{g.c.d.}(\hat{a}, b_i)$. There exists $\hat{Y_i}$, $\hat{X_i}$ such that

$$\hat{Y_i}\hat{a_i} + \hat{X_i}\hat{b_i} \equiv 1 \mod \ell \prod b_i$$

so that the matrix

$$\hat{\gamma_i} = \left(\begin{smallmatrix} \hat{a_i} & \hat{X_i}\hat{b} \\ \hat{c}\hat{b_i} & \hat{g_i}\hat{d} - \hat{Y_i}\hat{b}\hat{c} \end{smallmatrix}\right)$$

satisfies

$$\hat{\gamma_i}^{-1} \sigma_{b_i} \hat{\gamma}_{\mathfrak{a}} = \hat{\sigma_{b_i}} = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right).$$

Similarly, we decompose $g_i \hat{a} + Y_i b \hat{c} = \hat{A_i}\hat{g_i}', \hat{c} b_i = \hat{C_i}\hat{g_i}'$ so that there exists a matrix

$$\hat{\gamma_i}' = \left(\begin{smallmatrix} \hat{A_i} & * \\ \hat{C_i} & * \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}/\ell \prod b_i \mathbb{Z})$$

satisfying

$$\hat{\gamma_i}'^{-1} \sigma_{b_i'} \hat{\gamma}_{\mathfrak{a}} = \hat{\sigma_{b_i}}' = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right).$$

We let

$$\begin{aligned} \hat{T} x_i &= x_i \hat{\gamma_i} = (\hat{a_i} x_{i_1} + \hat{c}\hat{b_i} x_{i_2}, *) \\ \hat{T} x_i' &= T x_i \hat{\gamma_i}' = (\hat{A_i} T x_{i_1} + \hat{C_i} T x_{i_2}, *) \end{aligned}$$

so that

$$h^{\hat{\gamma}_{\mathfrak{a}}} = \prod g_{\hat{T}x_i}(\hat{\sigma_{b_i}}(L)/\mathbf{N}(\hat{\sigma}_{b_i}))$$
$$(h^{\sigma})^{\hat{\gamma}_{\mathfrak{a}}} = \prod g_{\hat{T}x'_i}(\hat{\sigma}'_{b_i}(L)/\mathbf{N}(\hat{\sigma}'_{b_i})).$$

Thus, we find

(7.18)

$$\begin{aligned}
&\mathrm{ord}_{\mathfrak{a}} h - \mathrm{ord}_{\mathfrak{a}} h^{\sigma} \\
&= \mathrm{ord}_{\infty} h^{\gamma_{\mathfrak{a}}} - \mathrm{ord}_{\infty} h^{\sigma\gamma_{\mathfrak{a}}} \\
&= 12\ell^2 \prod b_i \Big(\sum \frac{\hat{g}_i^2}{2b_i} \mathbf{B}_2(\hat{T}x_{i_1}) n_i - \frac{\hat{g}_i'^2}{2b_i} \mathbf{B}_2(\hat{T}x'_{i_1}) n_i \Big) \\
&\equiv \sum_i 6n_i \prod_{j\neq i} b_j ((r_1 s_2 b_i \hat{a})^2 + (r_2 s_1 b_i \hat{c})^2) - 6n_i \prod_{j \neq i} b_j \ell^2 \hat{g}_i'^2 \mathbf{B}_2(\hat{A}Tx_{i_1} + \hat{C}Tx_{i_1}) \\
&\equiv \sum_i 6n_i \prod_{j\neq i} b_j ((r_1 s_2 b_i \hat{a})^2 + (r_2 s_1 b_i \hat{c})^2) \\
&\qquad - 6n_i \prod_{j\neq i} b_j ((\hat{A}\hat{g}_i' \ell T x_{i_1})^2 + (\hat{C}\hat{g}_i' \ell T x_{i_2})^2 \\
&\qquad\qquad + 2\hat{A}\hat{C}\hat{g}_i'^2 T x_{i_1} T x_{i_2} - \hat{g}_i'^2 \hat{A} T x_{i_1} - \hat{g}_i'^2 \hat{C} T x_{i_2}) + \prod_{j\neq i} b_j n_i \ell^2 \hat{g}_i'^2 \\
&\equiv \sum_i 6n_i \prod_{j\neq i} b_j (((r_1 s_2 b_i \hat{a})^2 + (r_2 s_1 b_i \hat{c})^2) - ((\hat{A}\hat{g}_i')^2 ((r_1 s_2 a_i b_i)^2 + (r_2 s_1 c b'_i)^2) \\
&\qquad + (\hat{C}\hat{g}_i')^2 ((\delta X_i b r_1 s_2 b_i)^2 + (\delta(g_i d - Y_i bc) r_2 s_1)^2)) \\
&\qquad\qquad + 2\hat{A}\hat{C}\hat{g}_i'^2 ((r_1 b_i s_2)^2 a_i X_i b\delta + (r_2 s_1)^2 b'_i c\delta(g_i d - Y_i bc))) \\
&\equiv \sum_i 6n_i \prod_{j\neq i} b_j ((r_1 s_2 b_i \hat{a})^2 + (r_2 s_1 b_i \hat{c})^2) - ((g_i \hat{a} + Y_i b\hat{c})^2 (r_1 s_2 a b'_i)^2 \\
&\qquad + (\hat{c} b'_i)^2 (\delta r_2 s_1 g_i d)^2) \\
&\equiv \sum_i 6n_i \prod_{j\neq i} b_i ((\hat{a} b_i r_1 s_2)^2 + (\hat{c} b_i s_1 r_2)^2) - ((\hat{a} b_i r_1 s_2)^2 + (\hat{c} b_i s_1 r_2 d\delta)^2) \\
&\equiv \sum_i 6n_i \prod_{j\neq i} b_i ((\hat{a} b_i r_1 s_2)^2 (1 - (d\delta)^2)) \\
&\equiv 6 \prod_j b_j (r_2 s_1 \hat{c})^2 (1 - (d\delta)^2) \sum_i n_i b_i
\end{aligned}$$

where the congruence is taken mod ℓ. Now for $\mathfrak{a} = 0$, we let

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \hat{\gamma_i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\hat{\sigma_{b_i}} = \begin{pmatrix} b_i & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{\sigma_{b_i}}' = \begin{pmatrix} -b_i' & 0 \\ 0 & -g_i \end{pmatrix}.$$

Following the above argument in this case with $\hat{g_i} = 1, \hat{C_i} = 1, \hat{b_i} = b_i, \hat{a_i}' = 0$, and $\hat{c} = -1$, we find

$$\text{ord}_0 h - \text{ord}_0 h^{\sigma} \equiv 6 \prod_j b_j (r_2 s_1)^2 (1 - (d\delta)^2) \sum_i n_i b_i. \tag{7.19}$$

Assuming this sum is congruent to $0 \bmod \ell$, with hypotheses as stated on ℓ, b_i and n_i, we must have $s_2 \mid (d\delta)^2 - 1$. Thus we may conclude that $\text{Gal}(\mathcal{F}/\mathcal{F}_1(h)) \subseteq S_h$. By (7.18), we find $S_h \subseteq \text{Gal}(\mathcal{F}/\mathcal{F}_1(h))$, which proves Proposition 7.5. □

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