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## Configurations of rank- $40r$ extremal even unimodular lattices ( $r = 1, 2, 3$ )

par SCOTT DUKE KOMINERS et ZACHARY ABEL

RÉSUMÉ. Nous montrons que, si  $L$  est un réseau unimodulaire pair extrémal de rang  $40r$  avec  $r = 1, 2, 3$ , alors  $L$  est engendré par ses vecteurs de normes  $4r$  et  $4r + 2$ . Notre résultat est une extension de celui d'Ozeki pour le cas  $r = 1$ .

ABSTRACT. We show that if  $L$  is an extremal even unimodular lattice of rank  $40r$  with  $r = 1, 2, 3$ , then  $L$  is generated by its vectors of norms  $4r$  and  $4r + 2$ . Our result is an extension of Ozeki's result for the case  $r = 1$ .

### 1. Introduction

A lattice of rank  $n$  is a free  $\mathbb{Z}$ -module of rank  $n$  equipped with a positive-definite inner product  $(\cdot, \cdot) : L \times L \rightarrow \mathbb{R}$ . The dual of  $L$ , denoted  $L^*$ , is the set

$$L^* = \{y \in L \otimes \mathbb{R} : \forall x \in L, (x, y) \in \mathbb{Z}\},$$

which itself forms a lattice of the same rank as  $L$ . For a lattice vector  $x \in L$ , we call  $(x, x)$  the *norm* of  $x$ . A lattice  $L$  is *integral* if  $(x, x') \in \mathbb{Z}$  for all  $x, x' \in L$ , i.e. if and only if  $L \subseteq L^*$ . An integral lattice is said to be *unimodular* if it is self-dual ( $L = L^*$ ).

A lattice  $L$  is called *even* if and only if every lattice vector has an even integer norm, i.e.  $(x, x) \in 2\mathbb{Z}$  for  $x \in L$ . An even lattice is automatically integral by the familiar parallelogram identity,  $2(x, x') = (x + x', x + x') - (x, x) - (x', x')$ .

Lattices that are simultaneously even and unimodular are especially rare. Indeed, such a lattice's rank must be divisible by 8. Sloane proved that if  $L$  is an even unimodular lattice of rank  $n$  then the minimal (nonzero) norm in  $L$  is bounded by

$$(1.1) \quad \min_{\substack{x \in L \\ x \neq 0}} (x, x) \leq 2\lfloor n/24 \rfloor + 2$$

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*Mots clefs.* Even unimodular lattices, extremal lattices, weighted theta series.

(see [2, p. 194, Cor. 21]). An even unimodular lattice of rank  $n$  is called *extremal* if it attains the bound (1.1).

Ozeki [6, 8] showed that if  $L$  is an extremal even unimodular lattice of rank 32 or 48 then  $L$  is generated by its vectors of minimal norm. The first author [5] showed analogous results for extremal even unimodular lattices of ranks 56, 72, and 96. In a similar vein, Ozeki [7] showed that if  $L$  is extremal even unimodular of rank 40, then  $L$  is generated by its vectors of norms 4 and 6. Here, we extend and slightly simplify Ozeki’s methods, recovering Ozeki’s rank-40 result and obtaining analogous results for extremal even unimodular lattices of ranks 80 and 120.

### 2. Modular forms and theta series

We will use the notation  $\mathcal{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$  for the *upper half plane* of complex numbers. A *modular form of weight  $k$  for the group  $PSL_2(\mathbb{Z})$*  is a holomorphic function  $f : \mathcal{H} \rightarrow \mathbb{C}$  which is holomorphic at  $i\infty$  and satisfies

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$ . If a modular form  $f$  vanishes at  $z = i\infty$ , it is called a *cuspidal form*.

Let  $M_k$  and  $M_k^0$  be the  $\mathbb{C}$ -vector spaces of modular forms and cuspidal forms of weight  $k$  respectively. It is known that the *Eisenstein series*

$$E_4(z) = 1 + 240e^{2\pi iz} + 2160e^{4\pi iz} + 6720e^{6\pi iz} + \dots \text{ and}$$

$$E_6(z) = 1 - 504e^{2\pi iz} - 16632e^{4\pi iz} - 122976e^{6\pi iz} - \dots,$$

which are modular forms of weights 4 and 6 respectively, freely generate the spaces  $M_k$  in the sense that any nonzero modular form can be written uniquely as a weighted homogeneous polynomial in  $E_4$  and  $E_6$ . This implies that  $\dim(M_k) = 0$  for  $k$  odd, negative, or  $k = 2$ ; that  $\dim(M_{2k}) = 1$  and  $\dim(M_{2k}^0) = 0$  for  $k = 0, 2 \leq k \leq 5$  and  $k = 7$ ; and that multiplication by the weight-12 modular form  $\Delta = 12^{-3}(E_4^3 - E_6^2)$  defines an isomorphism  $M_{k-12} \xrightarrow{\sim} M_k^0$ . More information on the theory of modular forms for  $PSL_2(\mathbb{Z})$  can be found in [9].

The *theta function*  $\Theta_L : \mathcal{H} \rightarrow \mathbb{C}$  associated to a lattice  $L$  is defined by

$$\Theta_L(z) = \sum_{x \in L} e^{\pi i(x,x)z};$$

it is a generating function encoding the norms of  $L$ ’s vectors. For a homogeneous *harmonic polynomial*  $P \in \mathbb{C}[x_1, \dots, x_n]$ , i.e. a homogeneous polynomial for which  $\sum_{j=1}^n \frac{\partial^2 P}{\partial x_j^2} \equiv 0$ , we define the *weighted theta series*  $\Theta_{L,P}$  by

$$\Theta_{L,P}(z) = \sum_{x \in L} P(x)e^{\pi i(x,x)z}.$$

As shown in [9, 3], if  $L$  is an even unimodular lattice of rank  $n$  then  $\Theta_L$  is a modular form of weight  $\frac{n}{2}$ , and if in addition  $P$  is a homogeneous harmonic polynomial of degree  $d$ , then  $\Theta_{L,P}$  is a modular form of weight  $\frac{n}{2} + d$ .

### 3. Main result

We denote by  $P_{d,x_0}(x)$  the “zonal spherical harmonic polynomial” of degree  $d$ , related to the Gegenbauer polynomial by

$$(3.1) \quad P_{d,x_0}(x) = G_d((x, x_0), ((x, x)(x_0, x_0))^{1/2}),$$

where  $G_d(\cdot, \cdot)$  is the homogeneous polynomial of degree  $d$  such that  $G_d(t, 1)$  is the Gegenbauer polynomial of degree  $d$  evaluated at  $t$  [1].

We let  $L$  be an extremal even unimodular lattice of rank  $40r$  (where  $r \in \{1, 2, 3\}$ ), and adopt the notation used by Ozeki in [7]: For an even unimodular lattice  $L$ , we denote by  $\Lambda_{2m}(L)$  the set of vectors in  $L$  having norm  $2m$ . We denote by  $\mathcal{L}_{2m}(L)$  the sublattice of  $L$  generated by  $\Lambda_{2m}(L)$ , and similarly denote by  $\mathcal{L}_{2m_1+2m_2}(L)$  the sublattice of  $L$  generated by  $\Lambda_{2m_1}(L) \cup \Lambda_{2m_2}(L)$ .

We define  $a(2k, L) = |\Lambda_{2k}(L)|$ . It is clear that the theta series  $\Theta_L$  is given by  $\Theta_L(z) = \sum_{k=0}^{\infty} a(2k, L)e^{2k\pi iz}$ . We note that

$$4r = 2\lfloor 5r/3 \rfloor + 2 = \min\{2k > 0 : a(2k, L) \neq 0\}$$

is the minimal norm of vectors in  $L$  and use the notation

$$N_j(x) = |\{y \in \Lambda_{4r}(L) : (x, y) = j\}|,$$

$$M_j(x) = |\{y \in \Lambda_{4r+2}(L) : (x, y) = j\}|.$$

Using the involution  $y \longleftrightarrow -y$  of  $\Lambda_m(L)$ , we see that we have  $N_j(x) = N_{-j}(x)$  and  $M_j(x) = M_{-j}(x)$  for any  $j \in \mathbb{R}$  and  $x \in L \otimes \mathbb{R}$ .

We will show the following configuration result, which directly extends Ozeki’s [7] result for extremal even unimodular lattices of rank 40:

**Theorem 3.1.** *For  $r = 1, 2, 3$  and  $L$  extremal even unimodular of rank  $40r$ , we have  $L = \mathcal{L}_{4r+(4r+2)}(L)$ .*

*Proof.* We partition  $L$  into its equivalence classes modulo  $\mathcal{L}_{4r+(4r+2)}(L)$ . We need only show that any class  $[x] \in L/\mathcal{L}_{4r+(4r+2)}(L)$  is represented by a vector  $x_0 \in [x]$  with  $(x_0, x_0) \leq 4r + 2$ .

Now, we suppose there exists some equivalence class  $[x_0] \in L/\mathcal{L}_{4r+(4r+2)}(L)$  where  $x_0 \neq 0$  is a representative of minimal norm with  $(x_0, x_0) = 2t$  for some  $t \geq 2r + 2$ . We have the inequality

$$(3.2) \quad |(x_0, x)| \leq 2r \text{ for all } x \in \Lambda_{4r}(L).$$

Indeed, if  $(x_0, \pm x) > 2r$ , then  $L$  contains a vector  $x \mp x_0$  with norm

$$(x \mp x_0, x \mp x_0) = (x, x) \mp 2(x, x_0) + (x_0, x_0) < (x_0, x_0),$$

contradicting the minimality of  $x_0$ .

Similarly, we have

$$(3.3) \quad |(x_0, x)| \leq 2r + 1 \text{ for all } x \in \Lambda_{4r+2}(L).$$

From (3.2) and (3.3), we have the equations

$$(3.4) \quad \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2k} = \sum_{j=1}^{2r} 2 \cdot j^{2k} \cdot N_j(x_0),$$

$$(3.5) \quad \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2k} = \sum_{j=1}^{2r+1} 2 \cdot j^{2k} \cdot M_j(x_0),$$

for all  $k > 0$ .

We extract from the theta series  $\Theta_L$  of  $L$  the coefficients  $a(4r, L)$  and  $a(4r + 2, L)$ . We observe immediately from (3.4) and (3.5) that

$$(3.6) \quad \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^0 = a(4r, L),$$

$$(3.7) \quad \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^0 = a(4r + 2, L).$$

Since  $L$  is even unimodular of rank  $40r$ , we have  $\Theta_{L, P_d, x_0} \in M_{20r+d}^0$  for any  $d > 0$ . By comparing power-series coefficients, we then observe

$$(3.8) \quad \Theta_{L, P_d, x_0} \equiv 0 \text{ for } d \in \{2, \dots, 4r - 2, 4r + 2\},$$

$$(3.9) \quad \Theta_{L, P_{4r}, x_0} \equiv c_1 \Delta^{2r} \text{ for a constant } c_1,$$

$$(3.10) \quad \Theta_{L, P_{4r+4}, x_0} \equiv c_2 E_4 \Delta^{2r} \text{ for a constant } c_2.$$

From (3.8), we obtain the equations

$$(3.11) \quad \sum_{x \in \Lambda_{4r}(L)} (x, x_0)^{2d} = a(4r, L) \frac{1 \cdot 3 \cdots (2d - 1)}{40r \cdot (40r + 2) \cdots (40r + 2d - 2)} (8r)^d t^d$$

and

$$(3.12) \quad \sum_{x \in \Lambda_{4r+2}(L)} (x, x_0)^{2d} = a(4r + 2, L) \frac{1 \cdot 3 \cdots (2d - 1)}{40r \cdot (40r + 2) \cdots (40r + 2d - 2)} (8r + 4)^d t^d,$$

for  $d \in \{2, \dots, 4r - 2, 4r + 2\}$ . We obtain from (3.9)

$$(3.13) \quad \sum_{x \in \Lambda_{4r+2}(L)} P_{4r,x_0}(x) = c_{4r} \sum_{x \in \Lambda_{4r}(L)} P_{4r,x_0}(x),$$

where  $\Delta^{4r} = e^{(4r)\pi iz} + c_{4r}e^{(4r+1)\pi iz} + O(e^{(4r+2)\pi iz})$ . Similarly, (3.10) gives

$$(3.14) \quad \sum_{x \in \Lambda_{4r+2}(L)} P_{4r+4,x_0}(x) = c_{4r+4} \sum_{x \in \Lambda_{4r}(L)} P_{4r+4,x_0}(x),$$

where  $E_4\Delta^{4r} = e^{(4r)\pi iz} + c_{4r+4}e^{(4r+1)\pi iz} + O(e^{(4r+2)\pi iz})$ .

Combining the equations (3.6), (3.7), (3.11), (3.12), (3.13), and (3.14) with (3.4) and (3.5), we obtain a system of  $4r + 4$  homogeneous linear equations in the  $4r + 3$  unknowns

$$N_0(x_0), \dots, N_{2r}(x_0), M_0(x_0), \dots, M_{2r+1}(x_0).$$

At this stage, we diverge from our natural generalization of Ozeki's original methods and obtain the (extended) determinants of these inhomogeneous linear systems; these determinants must vanish because the system is overdetermined.

For  $r = 1, 2, 3$ , these determinants are respectively

$$(3.15) \quad 2^{55}3^75^87^411^413^119^623^3 \cdot (t - 2) \cdot t \cdot (6t - 13) \cdot (10t^2 - 55t + 77),$$

$$(3.16) \quad 2^{132}3^{27}5^{16}7^{10}11^613^{10}23^441^843^647^3 \cdot (t - 4) \cdot t \cdot Q_2(t),$$

$$(3.17) \quad 2^{244}3^{48}5^{26}7^{13}11^713^717^623^431^{11}37^{15}59^{14}61^{11}67^571^373^1 \cdot (t - 6) \cdot t \cdot Q_3(t),$$

where  $Q_2(t)$  is the irreducible quintic

$$10768t^5 - 242280t^4 + 2202310t^3 - 10101795t^2 + 23361877t - 21771246$$

and  $Q_3(t)$  is the irreducible septic

$$\begin{aligned} &19989882674056909935t^7 - 892881426107875310430t^6 \\ &+ 17258039601222654151533t^5 - 187053310321121904306075t^4 \\ &+ 1227398249908229181423784t^3 - 4874010945909263810320032t^2 \\ &+ 10840974078436271024624064t - 10414527769923133690990080. \end{aligned}$$

In each case, there are no integer solutions  $t \geq 2r + 2$ . However, we had assumed the existence of an equivalence class

$$[x_0] \in L/\mathcal{L}_{4r+(4r+2)}(L)$$

with minimal-norm representative  $x_0 \neq 0$  having  $(x_0, x_0) = 2t$  for integral  $t \geq 2r + 2$ ; since no such  $t$  exists, all equivalence classes must be generated by vectors having norms  $4r$  and  $4r + 2$ . □

#### 4. Concluding remarks

A quick inspection will show that our results are the only possible immediate extensions of Ozeki's methods. In the cases  $r \geq 4$ , it is not possible to extract sufficiently many linear conditions by these exact techniques, as the dimensions of the relevant spaces of cusp forms grow too large.

However, using different analysis, Elkies [4] has shown a stronger result than our Theorem 3.1 in the  $r = 3$  case: If  $L$  is an extremal even unimodular lattice of rank 120 then  $L = \mathcal{L}_{12}(L)$ . This result for rank-120 lattices is analogous to Ozeki's [6, 8] results in dimensions 32 and 48, and to the first author's [5] results in dimensions 56, 72, and 96.

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