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Résumé. Nous donnons une formule exacte pour la discrépance L_2 des ensembles généralisés de Zaremba, une sous-classe des ensembles plans généralisés de Hammersley en base b. Ces ensembles de points sont des décalés digitaux des ensembles de Hammersley obtenus avec un nombre arbitraire des différents décalages en base b. L'ensemble de Zaremba introduit par White en 1975 est le cas particulier où les b décalages possibles sont pris et répétés dans l'ordre, ce qui exige au moins b^b points pour atteindre la discrépance L_2 optimale. Au contraire, notre étude montre qu'il suffit d'un seul décalage non nul pour obtenir le même résultat, quelle que soit la base b.

Abstract. We give an exact formula for the L_2 discrepancy of a class of generalized two-dimensional Hammersley point sets in base b, namely generalized Zaremba point sets. These point sets are digitally shifted Hammersley point sets with an arbitrary number of different digital shifts in base b. The Zaremba point set introduced by White in 1975 is the special case where the b shifts are taken repeatedly in sequential order, hence needing at least b^b points to obtain the optimal order of L_2 discrepancy. On the contrary, our study shows that only one non-zero shift is enough for the same purpose, whatever the base b is.

1. Introduction and statement of the results

For a point set $\mathcal{P} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\}$ of $N \geq 1$ points in the two-dimensional unit-square $[0,1)^2$ the L_2 *discrepancy* is defined by

$$L_2(\mathcal{P}) := \left(\int_0^1 \int_0^1 |E(x,y,\mathcal{P})|^2 \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2},$$

where the so-called *discrepancy function* is given as $E(x,y,\mathcal{P}) = A([0,x) \times [0,y), \mathcal{P}) - Nxy$, where $A([0,x) \times [0,y), \mathcal{P})$ denotes the number of indices $1 \leq M \leq N$ for which $\boldsymbol{x}_M \in [0,x) \times [0,y)$. The L_2 discrepancy is a quantitative measure for the irregularity of distribution of $\mathcal{P}$, i.e., the deviation from ideal uniform distribution. See [5, 12] for more information.

It was first shown by Roth [16] that there is a constant $c > 0$ with the property that for the L_2 discrepancy of any finite point set $\mathcal{P}$ consisting of

N points in $[0,1)^2$ we have

$$L_2(\mathcal{P}) \geq c\sqrt{\log N}. \tag{1.1}$$

The first who obtained the order $\sqrt{\log N}$ for explicit point sets was Davenport [4] using what is now known as a process of symmetrization according to Proinov [15] (see also [1, 13]). More recently, in 2002, Chen and Skriganov [2, 3] provided a general construction in arbitrary dimension — a major breakthrough!— obtaining in the case of two dimensions the upper bound $C\sqrt{\log N}$ with $C = 11^4/(2\sqrt{11}) = 4727.43\ldots$. This approach (in the two-dimensional case) has a drawback: it only gives more or less loose upper bounds while other methods, using a careful analysis of the discrepancy function and initiated by Halton and Zaremba [10], allow for exact formulas, hence providing exact values for the leading constants implied in the O-notations. For example, like in the present study, Zaremba point sets give constants $C < 2$ for all bases $b \leq 50$ which is very low compared to the constant above obtained in base 11 by Chen and Skriganov.

In this paper we consider the L_2 discrepancy of generalizations in base b of the Halton-Zaremba point set in base 2, namely *generalized Zaremba point sets*. In that way, we follow the terminology of White [18] who extended the construction of Halton-Zaremba to arbitrary bases b and named the resulting point set *Zaremba point set*.

Generalized Zaremba point sets form a sub-class of generalized Hammersley point sets (see [7]) which we now define before to relate in detail the contributions of each other.

Throughout the paper the base $b \geq 2$ is an integer and $\mathfrak{S}_b$ is the set of all permutations of $\{0, \ldots, b-1\}$.

Definition (generalized Hammersley point set)**.** Let $b \geq 2$ and $n \geq 1$ be integers and let $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \mathfrak{S}_b^n$. For an integer $1 \leq N \leq b^n$, write $N - 1 = \sum_{r=0}^{n-1} a_r(N) b^r$ in the b-adic system and define $S_b^\Sigma(N) := \sum_{r=0}^{n-1} \frac{\sigma_r(a_r(N))}{b^{r+1}}$. Then the *generalized two-dimensional Hammersley point set in base* b consisting of b^n points associated to Σ is defined by

$$\mathcal{H}_{b,n}^\Sigma := \left\{ \left(S_b^\Sigma(N), \frac{N-1}{b^n} \right) \,:\, 1 \leq N \leq b^n \right\}.$$

If we choose in the above definition $\sigma_r = \mathrm{id}$ — the identity in $\mathfrak{S}_b$ — for all $r \in \{0, \ldots, n-1\}$, then we obtain the classical Hammersley point set in base b.

Exact formulas for the L_2 discrepancy of classical two-dimensional Hammersley point sets have been proved by Vilenkin [17], Halton and Zaremba [10] and Pillichshammer [14] in base $b = 2$ and by White [18] and Faure and Pillichshammer [7] for arbitrary bases. These results show that the classical Hammersley point set cannot achieve the best possible order of L_2 discrepancy with respect to Roth's general lower bound (1.1).

But their generalizations can do that and since the discovery by Halton and Zaremba [10] of a modification of the classical two-dimensional Hammersley point set in base 2 — which in fact was a special generalized Hammersley point set — achieving the best possible order, these generalizations have been studied by many authors.

Initial results were first available in base $b = 2$. At the beginning, of course, is the pioneering work of Halton and Zaremba [10]. Later their result has been recovered by Kritzer and Pillichshammer [11] who showed that for any $n \in \mathbb{N}$ and any $\Sigma \in \{\mathrm{id}, \mathrm{id}_1\}^n$, where $\mathrm{id}_1(k) := k + 1 \pmod 2$, we have

$$\left(L_2(\mathcal{H}^{\Sigma}_{2,n})\right)^2 = \frac{n^2}{64} - \frac{19n}{192} - \frac{ln}{16} + \frac{l^2}{16} + \frac{l}{4} + \frac{3}{8} + \frac{n}{16 \cdot 2^n} - \frac{l}{8 \cdot 2^n} + \frac{1}{4 \cdot 2^n} - \frac{1}{72 \cdot 4^n},$$

in which l is the number of id-permutations in Σ. It is remarkable that this formula only depends on the number of id-permutations in Σ and not on the distribution of them. For $l = \lfloor n/2 \rfloor$ the L_2 discrepancy is of order $O(n)$. (For more details we refer to [11]).

As to arbitrary bases it was first White [18] who generalized the result of Halton and Zaremba. Define special permutations by $\mathrm{id}_l(k) := k + l \pmod b$ for $0 \le l, k < b$ (the permutations id_l are called *digital shifts* in base b). Then White considered sequences Σ of the form

$$(\mathrm{id}_0, \mathrm{id}_1, \ldots, \mathrm{id}_{b-1}, \mathrm{id}_0, \mathrm{id}_1, \ldots, \mathrm{id}_{b-1}, \ldots) \tag{1.2}$$

of length n (White did not use this terminology) and he gave an exact formula for the L_2 discrepancy of the corresponding generalized Hammersley point set, which he named *Zaremba point set* as previously noticed. Essentially this formula states that

$$\left(L_2(\mathcal{H}^{\Sigma}_{b,n})\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1)$$

whenever Σ is of the form (1.2).

At this point, it should be remarked that another possibility exists to generalize the results of Halton and Zaremba (and of Kritzer and Pillichshammer respectively) to arbitrary bases, namely the use of a permutation τ defined by $\tau(k) = b - 1 - k$ which can be viewed as the mirror of the identity. Essentially, concatenating τ and id leads to the same leading constant but in more general situations [7]. Moreover, we have been able to generalize the same process to arbitrary permutations instead of identity only, which provides further improvements on leading constants. These results together with a lot of computational experiments are the topic of the paper [9].

Here we follow the approach of White but more generally we allow sequences of permutations $\Sigma \in \{\mathrm{id}_l : 0 \le l < b\}^n$, i.e., we do not demand the specific order of the permutations as in (1.2) neither the same number of each digital shift in base b. We call such a point set $\mathcal{H}^{\Sigma}_{b,n}$ with

$\Sigma \in \{\mathrm{id}_l : 0 \leq l < b\}^n$ a *generalized Zaremba point set* in base b with b^n points. For further work, notice that the same approach is possible with an arbitrary permutation instead of id, but more technical difficulties must be overcome before to reach results comparable to [9].

Now we state our main result:

Theorem 1.1. *Let $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\mathrm{id}_l : 0 \leq l < b\}^n$. For $0 \leq l < b$ define $\lambda_l := \#\{0 \leq i < n : \sigma_i = \mathrm{id}_l\}$ and, when $\lambda_l \neq 0$, for $1 \leq i \leq \lambda_l$ denote $h_i^{(l)}$ ($1 \leq h_i^{(l)} \leq n$) the integers such that $\sigma_{h_i^{(l)}-1} = \mathrm{id}_l$. Then we have*

$$\begin{aligned}\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = &\left(\sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right)^2 + n\frac{(b^2-1)(3b^2+13)}{720b^2} \\ &+ \left(1 - \frac{1}{2b^n}\right)\sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) + \frac{3}{8} + \frac{1}{4b^n} - \frac{1}{72b^{2n}} \\ &- \frac{1}{12b^{n+2}}\sum_{l=1}^{b-1}(b-l)l(b-2l)\sum_{i=1}^{\lambda_l}\left(b^{h_i^{(l)}} - b^{n-h_i^{(l)}+1}\right).\end{aligned}$$

The proof of this result will be given in Section 3.

For base $b = 2$ the above result recovers the result of Kritzer and Pillichshammer [11, Theorem 1] stated above.

If we choose $\lambda_0 = n$ and $\lambda_1 = \ldots = \lambda_{b-1} = 0$ then $\mathcal{H}_{b,n}^{\Sigma}$ is the classical Hammersley point set and our formula recovers [7, Theorem 1] and [18, Eq. (15)]. If we choose $\lambda_l = n$ for some $l \in \{0, \ldots, b-1\}$ and $\lambda_i = 0$ for all $i \neq l$, then we obtain the result from [8, Theorem 1] where we could consider a single shift only at the same time.

Remark 1. Note that we always have

$$\frac{1}{12b^{n+2}}\sum_{l=1}^{b-1}(b-l)l(b-2l)\sum_{i=1}^{\lambda_l}\left(b^{h_i^{(l)}} - b^{n-h_i^{(l)}+1}\right) = O(1)$$

with an implied constant only depending on b (and it is 0 for $b = 2$). Hence the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ with $\Sigma \in \{\mathrm{id}_l : 0 \leq l < b\}^n$ does mostly depend on the number of occurrences of id_l, $0 \leq l < b$, in Σ and almost not on the positions of them. For $b = 2$ the L_2 discrepancy of $\mathcal{H}_{2,n}^{\Sigma}$ only depends on the number of occurrences of id_0 and id_1 respectively in Σ and not on the positions of them. This is in accordance with [11, Remark 1].

The following corollary provides a choice of Σ which yields the best possible order of L_2 discrepancy.

Corollary 1.1. *Let* $\Sigma \in \{\mathrm{id}_l : 0 \le l < b\}^n$ *such that* $\lambda_l = \lfloor \frac{n}{b} \rfloor + \theta_l$ *with* $\theta_l \in \{0,1\}$ *for all* $0 \le l < b$. *Then we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1).$$

Proof. Since

$$\begin{aligned}\frac{1}{2b}\sum_{l=0}^{b-1}\lambda_l l(b-l) &= \frac{1}{2b}\sum_{l=0}^{b-1}\left(\left\lfloor\frac{n}{b}\right\rfloor+\theta_l\right)l(b-l)\\ &= \left\lfloor\frac{n}{b}\right\rfloor\frac{b^2-1}{12}+\frac{1}{2b}\sum_{l=0}^{b-1}\theta_l l(b-l)\\ &= n\frac{b^2-1}{12b}+O(1)\end{aligned}$$

and since $\sum_{l=0}^{b-1}\lambda_l = n$ it follows that

$$\sum_{l=0}^{b-1}\frac{\lambda_l}{b}\left(\frac{b^2-1}{12}-\frac{l(b-l)}{2}\right) = n\frac{b^2-1}{12b}-\frac{1}{2b}\sum_{l=0}^{b-1}\lambda_l l(b-l) = O(1).$$

Hence the result follows from Theorem 1.1 and Remark 1. □

Remark 2. Of course Corollary 1.1 contains the result of White with the sequence of shifts (1.2). One can also give the exact formula for the L_2 discrepancy of $\mathcal{H}_{b,n}^{\Sigma}$ for specific sequences Σ, for example for that one of White. But, as such exact formulas are very complicated, we do not state them here explicitly.

We want to remark further that there is a little inaccuracy in the result of White [18, Eq. (21)]. For example, he stated his result for the special case $b = 2$ in [18, Eq. (22)] which is *not* in accordance with the result of Kritzer and Pillichshammer [11, Theorem 1] that is recovered by our Theorem 1.1. White compares his formula with the result of Halton and Zaremba which shows slightly different expressions. However, as we know nowadays, for even n these formulas have to coincide.

We can even show that only one non-zero shift (together with id_0) is enough to get the best possible order of L_2 discrepancy.

Corollary 1.2. *Let* $\Sigma \in \{\mathrm{id}_l : 0 \le l < b\}^n$ *such that*

$$\lambda_0 = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } b = 2c+1,\\ \left\lceil n\frac{2c^2+1}{6c^2} \right\rceil & \text{if } b = 2c,\end{cases}$$

and

$$\lambda_c = \begin{cases} \left\lfloor \frac{2n}{3} \right\rfloor & \text{if } b = 2c+1,\\ \left\lfloor n\frac{4c^2-1}{6c^2} \right\rfloor & \text{if } b = 2c,\end{cases}$$

and $\lambda_l = 0$ *for* $l \notin \{0, c\}$. *Then we have*

$$\left(L_2(\mathcal{H}_{b,n}^{\Sigma})\right)^2 = n\frac{(b^2-1)(3b^2+13)}{720b^2} + O(1).$$

Proof. According to Theorem 1.1 and Remark 1 we only need to show that

$$\sum_{l=0}^{b-1} \lambda_l \left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) = O(1).$$

Assume that $b = 2c+1$. Then we have

$$\left|\sum_{l=0}^{b-1} \lambda_l \left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right| = \left|\left\lceil \frac{n}{3} \right\rceil \frac{c(c+1)}{3} - \left\lfloor \frac{2n}{3} \right\rfloor \frac{c(c+1)}{6}\right| \leq \frac{c(c+1)}{3}.$$

For $b = 2c$ we obtain

$$\left|\sum_{l=0}^{b-1} \lambda_l \left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right| \leq \frac{c^2}{2}.$$

□

Remark 3. In White's paper [18], the same result needs the b shifts id_l (for $0 \leq l < b$) in sequential order (see (1.2)) while in our paper [8, Corollary 1] we only need a single shift id_l, but the result is only valid for a small — although infinite — set of bases b (satisfying the Pell-Fermat equation $b^2 - 3d^2 = -2$ with a suitable integer d and $l = (b \pm d)/2$). Thanks to Theorem 1.1 and Corollary 1.2, we now have the result for all other bases with two shifts, id_0 and id_c. The interest of such improvements is that, since the optimal order of L_2 discrepancy is obtained with sets of b^n points, with one or two shifts the property starts being valid for b or b^2 points whereas with b shifts it needs at least b^b points, which is far away from usual numbers of points in applications using digital shifts (for instance in quasi-Monte Carlo methods), even for small bases. Theorem 1.1 and its corollaries conclude our investigations on digital shifts of original Hammersley point sets as announced at the end of [8, Section 1].

The rest of the paper is organized as follows: in Section 2 we prepare different technical auxiliary results — essentially six lemmas — which will be used for the proof of Theorem 1.1. Section 3 then is devoted to the actual proof of Theorem 1.1. This section starts with a proof of a discrete version of Theorem 1.1 (see Lemma 3.1) which is based on an exact formula for the discrepancy function of generalized Hammersley point sets (see Lemma 2.2) and on several other results (Lemmas 2.1, 2.3, 2.4, 2.5). Based on this discrete result we provide then the detailed proof of Theorem 1.1.

As a guideline for the reader, before entering through the details, we briefly outline the organization of the actual proof of Theorem 1.1: Using the discretization (2.5) from Remark 4, which is obtained from the structure of Hammersley point sets, the L_2 norm of the discrepancy function is split into three sums Σ_1, Σ_2 and Σ_3. For the evaluation of Σ_1 we use the discrete version of the theorem from Lemma 3.1. The evaluation of Σ_3 is a matter of straightforward computations. However, the remaining sum Σ_2, needs a further very thorough analysis involving Lemma 2.6 and a trick already used in [7, 8, 9] to complete the last estimation (computation of $\Sigma_{4,1}$) which then finishes the proof.

2. Auxiliary results

In this section we provide the main tools for the proof of Theorem 1.1. For the sake of completeness, we give short hints for the proofs of lemmas concerned with shifts and already proved in [8]. The analysis of the L_2 discrepancy is based on special functions which have been first introduced by Faure in [6] and which are defined as follows.

For $\sigma \in \mathfrak{S}_b$ let $\mathcal{Z}_b^\sigma = (\sigma(0)/b, \sigma(1)/b, \dots, \sigma(b-1)/b)$. For $0 \le h < b$ and $x \in [(k-1)/b, k/b)$ where $k \in \{1, \dots, b\}$ we define

$$\varphi_{b,h}^\sigma(x) = \begin{cases} A([0, h/b); k; \mathcal{Z}_b^\sigma) - hx & \text{if } 0 \le h \le \sigma(k-1), \\ (b-h)x - A([h/b, 1); k; \mathcal{Z}_b^\sigma) & \text{if } \sigma(k-1) < h < b, \end{cases}$$

where, here for a sequence $X = (x_M)_{M \ge 1}$, we denote by $A(I; k; X)$ the number of indices $1 \le M \le k$ such that $x_M \in I$. Further, the function $\varphi_{b,h}^\sigma$ is extended to the reals by periodicity. Note that $\varphi_{b,0}^\sigma = 0$ for any σ and that $\varphi_{b,h}^\sigma(0) = 0$ for any $\sigma \in \mathfrak{S}_b$ and any $0 \le h < b$.

For $r \in \mathbb{N}$ define $\varphi_b^{\sigma,(r)} := \sum_{h=0}^{b-1} (\varphi_{b,h}^\sigma)^r$ and we simply write $\varphi_b^\sigma := \varphi_b^{\sigma,(1)}$. Note that φ_b^σ is continuous, piecewise linear on the intervals $[k/b, (k+1)/b]$ and $\varphi_b^\sigma(0) = \varphi_b^\sigma(1)$.

For example, for $\sigma = \mathrm{id}$ we have

$$\varphi_{b,h}^{\mathrm{id}}(x) = \begin{cases} (b-h)x & \text{if } x \in [0, h/b], \\ h(1-x) & \text{if } x \in [h/b, 1], \end{cases} \tag{2.1}$$

from which one obtains (see [7, Lemma 3] for details) that for $x \in \left[\frac{k}{b}, \frac{k+1}{b}\right]$, $0 \le k < b$,

$$\varphi_b^{\mathrm{id}}(x) = \frac{b(b-2k-1)}{2}\left(x - \frac{k}{b}\right) + \frac{k(b-k)}{2}. \tag{2.2}$$

In order to deal with shifts of identity, we will use the following property from [1, Propriété 3.4] stating that

$$(\varphi_{b,h}^\sigma)'(k/b + 0) = (\varphi_{b,h}^{\mathrm{id}})'(\sigma(k)/b + 0). \tag{2.3}$$

The following lemma gives a relation between the $\varphi_{b,h}^\sigma$ functions with respect to the permutations id and id_l.

Lemma 2.1. *For any* $0 \le h, l < b$ *and* $x \in [0,1]$ *we have*

$$\varphi_{b,h}^{\mathrm{id}_l}(x) = \varphi_{b,h}^{\mathrm{id}}\left(x + \frac{l}{b}\right) - \varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right). \tag{2.4}$$

Proof. Since the functions $\varphi_{b,h}^{\sigma}$ are continuous and linear on $\left[\frac{k}{b}, \frac{k+1}{b}\right]$ with $k \in \{0,\ldots,b-1\}$, it is enough to show the equality for $x = k/b$ with $k \in \{0,\ldots,b-1\}$. Now, invoking (2.3) gives the desired result. (For details see [8, Proof of Lemma 1]). □

The following lemma provides a formula for the discrepancy function of arbitrary generalized Hammersley point sets.

Lemma 2.2. *For integers* $1 \le \lambda, N \le b^n$ *we have*

$$E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right) = \sum_{j=1}^{n} \varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right),$$

where the $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ *can be given explicitly.*

A proof of this result together with formulas for $\varepsilon_j = \varepsilon_j(\lambda, n, N)$ can be found in [7, Lemma 1]. It is the starting point of studies [7, 8, 9].

Remark 4. Let $0 \le x, y \le 1$ be arbitrary. Since all points from $\mathcal{H}_{b,n}^{\Sigma}$ have coordinates of the form α/b^n for some $\alpha \in \{0, 1, \ldots, b^n - 1\}$, we have

$$E(x, y, \mathcal{H}_{b,n}^{\Sigma}) = E(x(n), y(n), \mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n) - xy), \tag{2.5}$$

where for $0 \le x \le 1$ we define $x(n) := \min\{\alpha/b^n \ge x \,:\, \alpha \in \{0,\ldots,b^n\}\}$.

In the following we give a series of lemmas with further, more involved properties of the $\varphi_{b,h}^{\sigma}$ and $\varphi_b^{\sigma,(r)}$ functions.

Lemma 2.3. *For* $1 \le N \le b^n$, $0 \le j_1 < \ldots < j_k < n$ *and* $r_1, \ldots, r_k \in \mathbb{N}$, *we have*

$$\sum_{\lambda=1}^{b^n} \prod_{i=1}^{k} \left(\varphi_{b,\varepsilon_{j_i}}^{\sigma_{j_i}}\left(\frac{N}{b^{j_i}}\right)\right)^{r_i} = b^{n-k} \prod_{i=1}^{k} \varphi_b^{\sigma_{j_i},(r_i)}\left(\frac{N}{b^{j_i}}\right).$$

A proof of this result can be found in [7, Lemma 2].

The following lemma is a generalization of [8, Lemma 4], where the result is valid in case of $l = m$ only.

Lemma 2.4. *For* $0 \le h, k < n, h \ne k$ *and* $0 \le l, m < b$, *we have*

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^h}\right) \varphi_b^{\mathrm{id}_m}\left(\frac{N}{b^k}\right) = b^n \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\right)\left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}}\left(\frac{m}{b}\right)\right).$$

Proof. Using (2.4) from Lemma 2.1, we have

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^h}\right)\varphi_b^{\mathrm{id}_m}\left(\frac{N}{b^k}\right) =$$

$$= \sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{m}{b}\right) + b^n\varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\varphi_b^{\mathrm{id}}\left(\frac{m}{b}\right)$$

$$- \varphi_b^{\mathrm{id}}\left(\frac{m}{b}\right)\sum_{N=1}^{b^n}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right) - \varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\sum_{N=1}^{b^n}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{m}{b}\right). \tag{2.6}$$

From the periodicity of φ_b^{id} we obtain

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right) = b^{n-h}\sum_{N=0}^{b^h-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)$$

$$= b^{n-h}\sum_{N=0}^{b^{h-1}-1}\sum_{z=0}^{b-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{z}{b}\right) = b^n\frac{b^2-1}{12}, \tag{2.7}$$

since for fixed $0 \le N < b^{h-1}$ we have $\sum_{z=0}^{b-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{z}{b}\right) = b(b^2-1)/12$ as shown in [7, In proof of Lemma 5].

Without loss of generality we may assume that $h < k$. Then we have

$$\sum_{N=1}^{b^n} \varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{m}{b}\right)$$

$$= b^{n-k}\sum_{N=0}^{b^k-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{m}{b}\right)$$

$$= b^{n-k}\sum_{N=0}^{b^{k-1}-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)\sum_{z=0}^{b-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{z}{b}\right)$$

$$= \frac{b^2-1}{12}\sum_{N=0}^{b^n-1}\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right) = b^n\left(\frac{b^2-1}{12}\right)^2. \tag{2.8}$$

Now the result follows from inserting (2.7) and (2.8) into (2.6). □

The next two lemmas are proved in [8]. We just give some hints for the sake of completeness.

Lemma 2.5. *We have*

$$\sum_{N=1}^{b^n}\varphi_b^{\mathrm{id}_l,(2)}\left(\frac{N}{b^k}\right) = b^n\left(\frac{b^4-1}{90b}+\frac{b(b^2-1)}{36b^{2k}}\right) + b^n\varphi_b^{\mathrm{id},(2)}\left(\frac{l}{b}\right)$$

$$- \frac{b^{n-1}}{12}l(b-l)(1+b^2+lb-l^2).$$

Proof. Using Lemma 2.1, it is easy to deduce that

$$\varphi_b^{\mathrm{id}_l,(2)}\left(\frac{N}{b^k}\right) = \varphi_b^{\mathrm{id},(2)}\left(\frac{N}{b^k}+\frac{l}{b}\right)+\varphi_b^{\mathrm{id},(2)}\left(\frac{l}{b}\right)-2\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right).$$

Then, like in [7, Lemma 5, Part 2] one can show that

$$\sum_{N=1}^{b^n}\varphi_b^{\mathrm{id},(2)}\left(\frac{N}{b^k}+\frac{l}{b}\right)=b^n\left(\frac{b^4-1}{90b}+\frac{b(b^2-1)}{36b^{2k}}\right).$$

Finally, swapping the two sums and using (2.1) twice, the result follows since

$$\sum_{N=1}^{b^n}\sum_{h=0}^{b-1}\varphi_{b,h}^{\mathrm{id}}\left(\frac{N}{b^k}+\frac{l}{b}\right)\varphi_{b,h}^{\mathrm{id}}\left(\frac{l}{b}\right)=\frac{b^{n-1}}{24}l(b-l)(1+b^2+lb-l^2).$$

(For details see [8, Proof of Lemma 5].) □

Lemma 2.6. *For $0 \le h \le n$ and $0 \le l < b$ we have*

$$\sum_{N=1}^{b^n}N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)=b^{2n}\frac{b^2-1}{24}+\frac{b^n l(b-l)}{12b}(3b-b^h(b-2l)).$$

Proof. The idea is to split up the range of summation in order to use the periodicity of φ_b^{id} and its equation (2.2). After checking of intervals and values,

$$\sum_{N=1}^{b^n}N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}+\frac{l}{b}\right)=\sum_{r=0}^{b-1}\sum_{q=0}^{b^{n-h}-1}\sum_{N=qb^h+rb^{h-1}+1}^{qb^h+(r+1)b^{h-1}}N\varphi_b^{\mathrm{id}}\left(\frac{N}{b^h}-q+\frac{l}{b}\right),$$

which after splitting up of the first sum (from 0 to $b-l-1$ and $b-l$ to $b-1$) gives the desired result. (For details see [8, Proof of Lemma 6].) □

3. The proof of Theorem 1.1

First we show a discrete version of Theorem 1.1. The following result is a generalization of [8, Lemma 7] which can be obtained by choosing $\lambda_l = n$ for some l $(0 \le l < b)$ and $\lambda_i = 0$ for all $i \ne l$.

Lemma 3.1. *For $\Sigma = (\sigma_0, \ldots, \sigma_{n-1}) \in \{\mathrm{id}_l : 0 \le l < b\}^n$ let $\lambda_l := \#\{0 \le i < n : \sigma_i = \mathrm{id}_l\}$. Then we have*

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)=\sum_{l=0}^{b-1}\frac{\lambda_l}{b}\left(\frac{b^2-1}{12}-\frac{l(b-l)}{2}\right) \tag{3.1}$$

and

$$\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2=\left(\sum_{l=0}^{b-1}\frac{\lambda_l}{b}\left(\frac{b^2-1}{12}-\frac{l(b-l)}{2}\right)\right)^2$$
$$+\,n\frac{3b^4+10b^2-13}{720b^2}+\frac{1}{36}\left(1-\frac{1}{b^{2n}}\right). \tag{3.2}$$

Proof. We just give the (much more involved) proof of (3.2). Equation (3.1) can be shown in the same way.

Recall that when $\lambda_l\neq 0$ the integers $h_i^{(l)}$ introduced in the statement of Theorem 1.1 satisfy $\sigma_{h_1^{(l)}-1}=\ldots=\sigma_{h_{\lambda_l}^{(l)}-1}=\mathrm{id}_l$. Using Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned}
&\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2=\\
&=\frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\sum_{i,j=1}^{n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\\
&=\frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\left(\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\right)^2\\
&\quad+\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{N=1}^{b^n}\sum_{\lambda=1}^{b^n}\varphi_{b,\varepsilon_i}^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_{b,\varepsilon_j}^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\\
&=\frac{1}{b^{2n}}\sum_{i=1}^{n}\sum_{N=1}^{b^n}b^{n-1}\varphi_b^{\sigma_{i-1},(2)}\left(\frac{N}{b^i}\right)\\
&\quad+\frac{1}{b^{2n}}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\sum_{N=1}^{b^n}b^{n-2}\varphi_b^{\sigma_{i-1}}\left(\frac{N}{b^i}\right)\varphi_b^{\sigma_{j-1}}\left(\frac{N}{b^j}\right)\\
&=\frac{1}{b^{2n}}\sum_{l=0}^{b-1}\sum_{i=1}^{\lambda_l}\sum_{N=1}^{b^n}b^{n-1}\varphi_b^{\mathrm{id}_l,(2)}\left(\frac{N}{b^{h_i^{(l)}}}\right)\\
&\quad+\frac{1}{b^{2n}}\sum_{\substack{l,m=0\\ l\neq m}}^{b-1}\sum_{i=1}^{\lambda_l}\sum_{j=1}^{\lambda_m}\sum_{N=1}^{b^n}b^{n-2}\varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^{h_i^{(l)}}}\right)\varphi_b^{\mathrm{id}_m}\left(\frac{N}{b^{h_j^{m}}}\right)\\
&\quad+\frac{1}{b^{2n}}\sum_{l=0}^{b-1}\sum_{\substack{i,j=1\\ i\neq j}}^{\lambda_l}\sum_{N=1}^{b^n}b^{n-2}\varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^{h_i^{(l)}}}\right)\varphi_b^{\mathrm{id}_l}\left(\frac{N}{b^{h_j^{(l)}}}\right)\\
&=:A+B+C,
\end{aligned}$$

where in the penultimate equality, the second sum is simply omitted if $\lambda_l = 0$. Using Lemma 2.4 we get

$$B = \frac{1}{b^2} \sum_{\substack{l,m=0 \\ l \neq m}}^{b-1} \lambda_l \lambda_m \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}} \left(\frac{l}{b} \right) \right) \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}} \left(\frac{m}{b} \right) \right)$$

and

$$C = \frac{1}{b^2} \sum_{l=0}^{b-1} \lambda_l (\lambda_l - 1) \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}} \left(\frac{l}{b} \right) \right)^2 .$$

For A we use Lemma 2.5 and the fact that $\sum_{l=0}^{b-1} \lambda_l = n$ to obtain

$$\begin{aligned} A = {} & n \frac{b^4-1}{90b^2} + \frac{1}{36} \left(1 - \frac{1}{b^{2n}} \right) \\ & + \frac{1}{b} \sum_{l=0}^{b-1} \lambda_l \left(\varphi_b^{\mathrm{id},(2)} \left(\frac{l}{b} \right) - \frac{l(b-l)(1+b^2+lb-l^2)}{12b} \right) . \end{aligned}$$

Together we obtain

$$\begin{aligned} & \frac{1}{b^{2n}} \sum_{\lambda,N=1}^{b^n} \left(E \left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma} \right) \right)^2 \\ & \quad = \left(\sum_{l=0}^{b-1} \frac{\lambda_l}{b} \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}} \left(\frac{l}{b} \right) \right) \right)^2 - \sum_{l=0}^{b-1} \frac{\lambda_l}{b^2} \left(\frac{b^2-1}{12} - \varphi_b^{\mathrm{id}} \left(\frac{l}{b} \right) \right)^2 \\ & \qquad + n \frac{b^4-1}{90b^2} + \frac{1}{36} \left(1 - \frac{1}{b^{2n}} \right) \\ & \qquad + \sum_{l=0}^{b-1} \frac{\lambda_l}{b} \left(\varphi_b^{\mathrm{id},(2)} \left(\frac{l}{b} \right) - \frac{l(b-l)(1+b^2+lb-l^2)}{12b} \right) \\ & \quad = \left(\sum_{l=0}^{b-1} \frac{\lambda_l}{b} \left(\frac{b^2-1}{12} - \frac{l(b-l)}{2} \right) \right)^2 + n \frac{3b^4+10b^2-13}{720b^2} + \frac{1}{36} \left(1 - \frac{1}{b^{2n}} \right) \end{aligned}$$

where we used that $\varphi_b^{\mathrm{id}}(l/b) = l(b-l)/2$ and $\varphi_b^{\mathrm{id},(2)}(l/b) = (1-l/b)^2 l (l+1)(2l+1)/6 + (b-l)(b-l-1)(2b-2l-1)l^2/(6b^2)$ which follows from [7, Lemma 3]. □

Now we give the proof of Theorem 1.1.

Proof. Using (2.5) we obtain

$$
\begin{aligned}
\Big(L_2(\mathcal{H}_{b,n}^{\Sigma})\Big)^2 &= \int_0^1\int_0^1 \Big(E(x(n),y(n),\mathcal{H}_{b,n}^{\Sigma}) + b^n(x(n)y(n)-xy)\Big)^2\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{b^{2n}}\sum_{\lambda,N=1}^{b^n}\left(E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\right)^2 \\
&\quad + 2b^n\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}} E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right)\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)\mathrm{d}x\,\mathrm{d}y \\
&\quad + b^{2n}\sum_{\lambda,N=1}^{b^n}\int_{\frac{\lambda-1}{b^n}}^{\frac{\lambda}{b^n}}\int_{\frac{N-1}{b^n}}^{\frac{N}{b^n}}\left(\frac{\lambda}{b^n}\frac{N}{b^n}-xy\right)^2\mathrm{d}x\,\mathrm{d}y \\
&=: \Sigma_1+\Sigma_2+\Sigma_3.
\end{aligned}
$$

The term Σ_1 has been evaluated in Lemma 3.1 and straightforward calculus shows that $\Sigma_3 = (1+18b^n+25b^{2n})/(72b^{2n})$. So it remains to deal with Σ_2.

Evaluating the integral appearing in Σ_2 we obtain

$$
\begin{aligned}
\Sigma_2 &= \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n}(\lambda+N)E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) - \frac{1}{2b^{3n}}\sum_{\lambda,N=1}^{b^n}E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) \\
&=: \Sigma_4-\Sigma_5.
\end{aligned}
$$

The term Σ_5 is obtained from (3.1) in Lemma 3.1. For Σ_4 we have

$$
\begin{aligned}
\Sigma_4 &= \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n}\lambda E\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) + \frac{1}{b^{3n}}\sum_{\lambda,N=1}^{b^n}NE\left(\frac{\lambda}{b^n},\frac{N}{b^n},\mathcal{H}_{b,n}^{\Sigma}\right) \\
&=: \frac{1}{b^{3n}}(\Sigma_{4,1}+\Sigma_{4,2}).
\end{aligned}
$$

As to $\Sigma_{4,2}$, with Lemma 2.2, Lemma 2.3, Lemma 2.1 and Lemma 2.6 we obtain (the second sum is simply omitted if $\lambda_l = 0$)

$$
\begin{aligned}
\Sigma_{4,2} &= b^{n-1}\sum_{l=0}^{b-1}\sum_{i=1}^{\lambda_l}\sum_{N=1}^{b^n}N\left(\varphi_b^{\mathrm{id}}\left(\frac{N}{b^{h_i^{(l)}}}+\frac{l}{b}\right)-\varphi_b^{\mathrm{id}}\left(\frac{l}{b}\right)\right) \\
&= b^{2n-1}\sum_{l=0}^{b-1}\sum_{i=1}^{\lambda_l}\left(b^n\frac{b^2-1}{24}+l(b-l)\left(\frac{3b-b^{h_i^{(l)}+1}+2lb^{h_i^{(l)}}}{12b}-\frac{b^n+1}{4}\right)\right) \\
&= b^{3n}\frac{b^2-1}{24b}n-\frac{b^{2n}}{12b^2}\sum_{l=1}^{b-1}\sum_{i=1}^{\lambda_l}(b-l)l\left(b^{h_i^{(l)}}(b-2l)+3b^{n+1}\right),
\end{aligned}
$$

where again integers $h_i^{(l)}$ satisfy $\sigma_{h_1^{(l)}-1} = \ldots = \sigma_{h_{\lambda_l}^{(l)}-1} = \mathrm{id}_l$ if $\lambda_l \neq 0$.

We turn to $\Sigma_{4,1}$. Let $g : [0,1]^2 \to [0,1]^2$ be defined by $g(x,y) = (y,x)$ and for $\Sigma = (\sigma_0, \ldots, \sigma_{n-1})$ define $\Gamma = (\gamma_0, \ldots, \gamma_{n-1}) := (\sigma_{n-1}^{-1}, \ldots, \sigma_0^{-1})$. Then it is easy to see (for details see [7, Proof of Theorem 4]) that $\mathcal{H}_{b,n}^{\Sigma} = g\left(\mathcal{H}_{b,n}^{\Gamma}\right)$. Therefore we obtain

$$\begin{aligned}\Sigma_{4,1} &= \sum_{\lambda,N=1}^{b^n} \lambda E\left(\frac{\lambda}{b^n}, \frac{N}{b^n}, \mathcal{H}_{b,n}^{\Sigma}\right)\\ &= \sum_{\lambda,N=1}^{b^n} \lambda E\left(\frac{N}{b^n}, \frac{\lambda}{b^n}, \mathcal{H}_{b,n}^{\Gamma}\right),\end{aligned}$$

which will allow us to use the result for $\Sigma_{4,2}$. To this end, we must check the correspondences between Σ and Γ: For $\Sigma \in \{\mathrm{id}_l : 0 \le l < b\}^n$ we also have $\Gamma \in \{\mathrm{id}_l : 0 \le l < b\}^n$. If $l = 0$ we get $\mathrm{id}_0^{-1} = \mathrm{id}_0$ and for $0 < l < b$, we get $\mathrm{id}_l^{-1} = \mathrm{id}_{b-l}$. Hence for $l > 0$ and $\lambda_l \neq 0$ we have $\sigma^{-1}_{h_1^{(l)}-1} = \ldots = \sigma^{-1}_{h_{\lambda_l}^{(l)}-1} = \mathrm{id}_{b-l}$, so that $\gamma_{n+1-h_1^{(l)}-1} = \ldots = \gamma_{n+1-h_{\lambda_l}^{(l)}-1} = \mathrm{id}_{b-l}$. Further $\gamma_{u_1^{(b-r)}-1} = \ldots = \gamma_{u_{\lambda_{b-r}}^{(b-r)}-1} = \mathrm{id}_r$, where $u_i^{(b-r)} := n - h_i^{(b-r)} + 1$. Now we may use the formula for $\Sigma_{4,2}$ and obtain

$$\begin{aligned}\Sigma_{4,1} &= b^{3n}\frac{b^2-1}{24b}n - \frac{b^{2n}}{12b^2}\sum_{r=1}^{b-1}\sum_{i=1}^{\lambda_{b-r}}(b-r)r\left(b^{u_i^{(b-r)}}(b-2r) + 3b^{n+1}\right)\\ &= b^{3n}\frac{b^2-1}{24b}n - \frac{b^{2n}}{12b^2}\sum_{r=1}^{b-1}\sum_{i=1}^{\lambda_{b-r}}(b-r)r\left(b^{n-h_i^{(b-r)}+1}(b-2r) + 3b^{n+1}\right)\\ &= b^{3n}\frac{b^2-1}{24b}n - \frac{b^{2n}}{12b^2}\sum_{l=1}^{b-1}\sum_{i=1}^{\lambda_l}(b-l)l\left(b^{n-h_i^{(l)}+1}(2l-b) + 3b^{n+1}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}\Sigma_4 = \frac{b^2-1}{12b}n - \frac{1}{2b}\sum_{l=0}^{b-1}(b-l)l\lambda_l\\ - \frac{1}{12b^{n+2}}\sum_{l=1}^{b-1}(b-l)l(b-2l)\sum_{i=1}^{\lambda_l}\left(b^{h_i^{(l)}} - b^{n-h_i^{(l)}+1}\right).\end{aligned}$$

Now we obtain

$$\left(L_2(\mathcal{H}^{\Sigma}_{b,n})\right)^2 = \left(\sum_{l=0}^{b-1} \frac{\lambda_l}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right)\right)^2 + n\frac{3b^4+10b^2-13}{720b^2}$$
$$+ \frac{1}{36}\left(1-\frac{1}{b^{2n}}\right) + \frac{b^2-1}{12b}n - \frac{1}{2b}\sum_{l=0}^{b-1}(b-l)l\lambda_l$$
$$- \frac{1}{12b^{n+2}}\sum_{l=1}^{b-1}(b-l)l(b-2l)\sum_{i=1}^{\lambda_l}\left(b^{h_i^{(l)}} - b^{n-h_i^{(l)}+1}\right)$$
$$- \frac{1}{2b^n}\sum_{l=0}^{b-1}\frac{\lambda_l}{b}\left(\frac{b^2-1}{12} - \frac{l(b-l)}{2}\right) + \frac{1+18b^n+25b^{2n}}{72b^{2n}}$$

which yields the desired result. □

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