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The summatory function of q-additive functions on pseudo-polynomial sequences

par Manfred G. MADRITSCH

Résumé. Le présent article étudie la fonction sommatoire de fonctions définies sur les chiffres en base q. En particulier, si n est un entier positif, nous notons

$$n=\sum_{r=0}^{\ell} d_r(n)q^r \quad \text{avec} \quad d_r(n)\in\{0,\dots,q-1\}$$

son développement en base q. Nous disons qu'une fonction f est *strictement q-additive* si, pour une valeur donnée, elle agit uniquement sur les chiffres de sa représentation, *i.e.*,

$$f(n)=\sum_{r=0}^{\ell} f\left(d_r(n)\right).$$

Soit $p(x)=\alpha_0 x^{\beta_0}+\dots+\alpha_d x^{\beta_d}$ avec $\alpha_0,\alpha_1,\dots,\alpha_d,\in\mathbb{R}$, $\alpha_0>0$, $\beta_0>\dots>\beta_d\geq 1$ et au moins un $\beta_i\notin\mathbb{Z}$. Un tel p est appelé pseudo-polynôme.

Le but est de prouver que pour f une fonction q-additive, il existe un $\varepsilon>0$ tel que

$$\sum_{n\leq N} f\left(\lfloor p(n)\rfloor\right)=\mu_f N\log_q(p(N)) + NF_{f,\beta_0}\left(\log_q(p(N))\right)+\mathcal{O}\left(N^{1-\varepsilon}\right),$$

où μ_f est la moyenne des valeurs de f et F_{f,β_0} est une fonction 1-périodique dérivable nulle part.

Ce résulat est motivé par des résultats de Nakai et Shiokawa et de Peter.

Abstract. The present paper deals with the summatory function of functions acting on the digits of an q-ary expansion. In

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particular let n be a positive integer, then we call

$$n = \sum_{r=0}^{\ell} d_r(n) q^r \quad \text{with} \quad d_r(n) \in \{0, \dots, q-1\}$$

its q-ary expansion. We call a function f *strictly q-additive*, if for a given value, it acts only on the digits of its representation, *i.e.*,

$$f(n) = \sum_{r=0}^{\ell} f\left(d_r(n)\right).$$

Let $p(x) = \alpha_0 x^{\beta_0} + \dots + \alpha_d x^{\beta_d}$ with $\alpha_0, \alpha_1, \dots, \alpha_d, \in \mathbb{R}$, $\alpha_0 > 0$, $\beta_0 > \dots > \beta_d \geq 1$ and at least one $\beta_i \notin \mathbb{Z}$. Then we call p a pseudo-polynomial.

The goal is to prove that for a q-additive function f there exists an $\varepsilon > 0$ such that

$$\sum_{n \leq N} f\left(\lfloor p(n) \rfloor\right) = \mu_f N \log_q(p(N)) + N F_{f,\beta_0}\left(\log_q(p(N))\right) + \mathcal{O}\left(N^{1-\varepsilon}\right),$$

where μ_f is the mean of the values of f and F_{f,β_0} is a 1-periodic nowhere differentiable function.

This result is motivated by results of Nakai and Shiokawa and Peter.

1. Introduction

Let $q \geq 2$ be an integer. Then we can represent every positive integer n in a unique way as

$$n = \sum_{r=0}^{\ell} d_r(n) q^r \quad \text{with} \quad d_r(n) \in \{0, \dots, q-1\}. \tag{1.1}$$

We call this the *q-ary representation* of n with q the *base* and $\{0, \dots, q-1\}$ the *set of digits.*

If a function f acts only on the digits of a representation, *i.e.*,

$$f(n) = \sum_{r=0}^{\ell} f\left(d_r(n) q^r\right),$$

where n is as in (1.1), then we call it *q-additive.* If this action of f is independent of the position of the digit, *i.e.*,

$$f(n) = \sum_{r=0}^{\ell} f\left(d_r(n)\right),$$

then we call f *strictly q-additive.* In the following we will concentrate on this type of functions.

A simple example of a strictly q-additive function is the sum-of-digits function s_q which is defined by

$$s_q(n) = \sum_{r \geq 0} d_r(n),$$

where n is again as in (1.1). Strictly q-additive functions have been investigated from several points of view. In the present paper we want to concentrate on arithmetical properties and, in particular, on the summatory function of f on pseudo-polynomial values.

Before we present the result we want to give an overview on what is known for the summatory function in connection with the sum-of-digits function. One of the first results in that direction is due to Delange [1] who was able to show

$$\sum_{n \leq N} s_q(n) = \frac{q-1}{2} N \log_q N + NF\left(\log_q N\right),$$

where $\log_q$ is the logarithm to base q and F is a 1-periodic, continuous and nowhere differentiable function. Remarkable about this result is the lack of an error term.

The moments of the sum-of-digits function were considered by Kirschenhofer [8] and by Grabner *et al.* [6]. All the methods used in this paper are based on Fourier analysis and in particular on the estimation of exponential sums. For different methods originating from the analysis of algorithms such as Mellin's formula elegant proofs have been shown by Flajolet *et al.* [2] and Grabner and Hwang [5]. Generalizations to number systems in number fields were done by Thuswaldner [15] and Gittenberger and Thuswaldner [4].

Apart from the sequence of the positive integers others like the primes or the integer values of polynomials are of interest. For the case of primes the summatory function of the sum-of-digits function was investigated by Shiokawa [14] whereas Mauduit and Rivat considered the distribution in residue classes. They also investigated the uniform distribution modulo 1 of $(\alpha s_q(\lfloor n^c \rfloor))_{n\geq 1}$ (with α an irrational and $\lfloor \cdot \rfloor$ the floor function) with $1 \leq c \leq \frac{7}{5}$ in [10], and with $c = 2$ in [11]. The distribution for the case $c = 1$ goes back to Gelfond [3].

Similar ideas are needed in order to construct a normal number, where one is also interested in the distribution of the digits in an expansion. A normal number $x \in \mathbb{R}$ is, informally speaking, a number in whose digital expansion every block of length k occurs asymptotically equally often. In a paper by Nakai and Shiokawa [12] they constructed such a normal number by concatenating the integer part of a pseudo-polynomial sequence, *i.e.* a sequence $(\lfloor p(n) \rfloor)_{n\geq 1}$ where

$$p(x) = \alpha_0 x^{\beta_0} + \alpha_1 x^{\beta_1} + \cdots + \alpha_d x^{\beta_d} \tag{1.2}$$

with $\alpha_0, \beta_0, \dots, \alpha_d, \beta_d \in \mathbb{R}$, $\alpha_0 > 0$, $\beta_0 > \beta_1 > \dots > \beta_d \geq 1$ and at least one $\beta_i \notin \mathbb{Z}$. Since we will often use β_0 we will set $\beta := \beta_0$ for short. Nakai and Shiokawa gained the following result as a corollary.

Theorem ([12, Corollary 2]). *Let p be as in (1.2). Then*

$$\sum_{n \leq N} s_q(\lfloor p(n) \rfloor) = \frac{q-1}{2} N \log_q p(N) + \mathcal{O}(N)$$

where $\log_q$ denotes the logarithm to base q.

Now we draw our attention towards a recent result by Peter [13] who considered polynomial sequences of the form $\left(\left\lfloor \alpha n^k \right\rfloor\right)_{n \geq 1}$ where α is one or an irrational of finite type. In particular, he could show the following.

Theorem ([13, Theorem]). *Let $q, k \geq 2$ be integers, and $\alpha = 1$ or $\alpha > 0$ an irrational of finite type. There are $c \in \mathbb{R}$ and $\varepsilon > 0$ such that*

$$\sum_{n \leq N} s_q(\left\lfloor \alpha n^k \right\rfloor) = \frac{q-1}{2} N \log_q(\alpha N^k) + cN + NF\left(\log_q(\alpha N^k)\right) + \mathcal{O}(N^{1-\varepsilon}),$$

with $N \geq 1$ and $\log_q$ the logarithm to base q.

Our aim is now to combine the last two results and to generalize them to arbitrary q-additive functions instead of the sum-of-digits function.

In order to state the result in a more convenient way we need some definitions. For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the floor function and by $\{x\} := x - \lfloor x \rfloor$ the fractional part of x, respectively. Furthermore we denote by $\|x\|$ the distance of x to the nearest integer, *i.e.*, $\|x\| := \min(\{x\}, 1 - \{x\})$. Let ψ be the centralized fraction function defined by

$$\psi(x) = x - \lfloor x \rfloor - \frac{1}{2} = \{x\} - \frac{1}{2}. \tag{1.3}$$

Moreover, we define

(1.4)

$$\mu_f := \sum_{a=0}^{q-1} f(a)$$

$$J_{f,\beta}(x) := \int_0^x \sum_{a=0}^{q-1} f(a) \left(\psi\left(t - \frac{a+1}{q}\right) - \psi\left(t - \frac{a}{q}\right)\right) t^{\frac{1}{\beta}-1} \mathrm{d}t,$$

with $x \geq 0$, $q \in \mathbb{N}$, and

$$F_{f,\beta}(t) := \mu_f(1 - \{t\}) + \frac{1}{\beta} q^{(1-\{t\})/\beta} \sum_{n \geq 0} q^{-\frac{n}{\beta}} J_{f,\beta}(q^{n-1+\{t\}}), \quad t \in \mathbb{R}.$$

Now we are able to state our result.

Theorem 1.1. *Let $q \geq 2$ be an integer and f be a strictly q-additive function with $f(0) = 0$. If p is a pseudo-polynomial as defined in (1.2), then there exists $\varepsilon > 0$ such that*

(1.5)
$$\sum_{n \leq N} f\left(\lfloor p(n) \rfloor\right) = \mu_f N \log_q(p(N)) + N F_{f,\beta}\left(\log_q(p(N))\right) + \mathcal{O}\left(N^{1-\varepsilon}\right).$$

Remark 1.1. We can show a similar result if f is q-additive but not necessarily strictly q-additive. In this more general setting one has to keep track of the position of a digit in the q-ary expansion throughout the whole proof. This leads to a more delicate periodic function $F_{f,\beta}$ which then depends on the integer value of $\log_q(p(N))$ (*i.e.*, the position) too.

The idea of proof is to consider each digit of an expansion according to its position within the expansion. Then we will distinguish two ranges for the position which are treated in Sections 2 and 3 separately. In particular, we consider sums of the form

$$U_{r,a}(N) := \sum_{N < n \leq 2N} \psi\left(\frac{p(n)}{q^{r+1}} - \frac{a}{q}\right), \tag{1.6}$$

where $a \in \{0, \ldots, q-1\}$ (the digit) and r (the position) lies in

$$1 \leq q^{r+1} \leq N^{\beta-1} \quad \text{and} \tag{1.7}$$

$$N^{\beta-1} \leq q^{r+1} \leq p(2N), \tag{1.8}$$

respectively. Finally in Section 4 we apply those estimates in order prove Theorem 1.1.

2. Exponential sums

This section focuses on the estimation of (1.6) for r such that (1.7) holds. These estimates will provide us with the error term. In particular, we will show the following.

Proposition 2.1. *Let N be positive and r be an integer such that (1.7) holds. Then there exists $\sigma_1 > 0$ such that*

$$U_{r,a}(N) \ll N^{1-\sigma_1} \tag{2.1}$$

We will first use an idea of Vinogradov in order to calculate the Fourier transformation of ψ and thus to get some exponential sums which are easier to treat. Therefore we write for short $e(x) := \exp(2\pi i x)$. The following lemma will provide us with the Fourier transformation of $U_{r,a}$.

Lemma 2.1 ([9, Theorem 1.8]). *Let $g(t)$ be a real function in $[a,b]$. Then for $\delta > 0$ the estimation*

$$\sum_{a<n\leq b}\psi(g(n)) \ll \frac{b-a}{\delta} + \sum_{\nu=1}^{\infty}\min\left(\frac{\delta}{\nu^2},\frac{1}{\nu}\right)\left|\sum_{a<n\leq b}e(\nu g(n))\right|$$

holds.

The exponential sums arising in Lemma 2.1 are very similar to those in Nakai and Shiokawa [12]. Thus we will use their lemma for estimation.

Lemma 2.2 ([12, Lemma 6]). *Let k, P and N be integers such that $k \geq 2$, $2 \leq N \leq P$. Let $g(x)$ be real and have continuous derivatives up to the $(k+1)$th order in $[P+1, P+N]$; let $0 < \lambda < 1/(2c_0(k+1))$ and*

$$\lambda \leq \frac{g^{(k+1)}(x)}{(k+1)!} \leq c_0\lambda \quad (P+1 \leq x \leq P+N),$$

or the same for $-g^{(k+1)}(x)$, and let

$$N^{-k-1+\delta} \leq \lambda \leq N^{-1}$$

with $0 < \delta \leq k$. Then

$$\sum_{n=P+1}^{P+N} e(g(n)) \ll N^{1-\eta},$$

where

$$\eta = \frac{\delta}{16(k+1)L},\ L = 1+\left\lfloor\frac{1}{4}k(k+1)+kR\right\rfloor,\ R = 1+\left\lfloor\frac{\log\left(\frac{1}{\delta}k(k+1)^2\right)}{-\log\left(1-\frac{1}{k}\right)}\right\rfloor.$$

If β (the highest exponent) is an integer we need some more delicate tools. In particular, we need three lemmas. The first two will be useful in the adoption of Lemma 2.5.

Lemma 2.3 ([16, Lemma 6.11]). *Let M and N be integers, $N > 1$, and let $\phi(n)$ be a real function of n, defined for $M \leq n \leq M+N-1$, such that*

$$\delta \leq \phi(n+1)-\phi(n) \leq c\delta \quad (M \leq n \leq M+N-2),$$

where $\delta > 0$, $c \geq 1$, $c\delta \leq \frac{1}{2}$. Let $W \geq 1$, then the number of values of n for which $\|\phi(n)\| \leq W\delta$ is less than

$$(Nc\delta+1)(2W+1).$$

We also state the Vinogradov integral, which will provide us with the desired estimate for the case of β being an integer. Let d and m be positive integers. We set

$$S(m) = \sum_{n=1}^{m} e(t_1 n + t_2 n^2 + \cdots + t_d n^d),$$

where $t_1, \ldots, t_d$ are reals. Then for an integer L we set

$$J(m, L) = \int_0^1 \cdots \int_0^1 |S(m)|^{2L} \, \mathrm{d}t_1 \cdots \mathrm{d}t_d.$$

Now we can state the estimate of the Vinogradov integral.

Lemma 2.4 ([16, Lemma 6.9])**.** *If R is any non-negative integer and $L \geq \frac{1}{4}d(d+1) + dR$, then*

$$J(m, L) \leq K^R (\log m)^R m^{2L - \frac{1}{2}d(d+1) + \frac{1}{2}d(d+1)(1-(1/d))^R},$$

where $K = 48^{2L}(L!)^2 L^d d^{\frac{1}{2}d(d-1)}$.

The last lemma deals with a variation of Lemma 2.2 for twice differentiable functions.

Lemma 2.5 ([7, Corollary 8.12])**.** *Let $f(x)$ be a real function on $[a, b]$, twice differentiable, and let $f''(x) \geq \lambda > 0$ throughout the interval (a, b). Then*

$$\sum_{a < n \leq b} e(f(n)) \ll (f'(b) - f'(a) + 1)\lambda^{-\frac{1}{2}}.$$

What is left is the keystone in the proof of Proposition 2.1. In particular, in the proof we will have to estimate exponential sums in a uniform way, which is provided by the following lemma.

Lemma 2.6. *Let N be a positive real and p is a pseudo-polynomial as in (1.2) with $d \geq 1$. If γ and r are such that*

$$0 < \gamma < \min\left(\frac{\beta - \beta_1}{2}, \frac{1}{3}\right) \quad \text{and} \quad 1 \leq q^{r+1} \leq N^{\beta - 1} \tag{2.2}$$

hold then there exists an $\eta > 0$ such that for $1 \leq \nu \leq N^{\gamma}$ we have

$$S(N, r, \nu) = \sum_{N < n \leq 2N} e\left(\frac{\nu p(n)}{q^{r+1}}\right) \ll N^{1-\eta}.$$

Proof. In order to properly estimate $S(N, r, \nu)$ we divide the proof up into two cases, according to whether $\beta \notin \mathbb{Z}$ or $\beta \in \mathbb{Z}$.

- **Case 1: β is not an integer.** We put $k := \lfloor \beta \rfloor + 2$ and $g(x) := \nu q^{-(r+1)} p(x)$. The idea is an application of Lemma 2.2. Therefore we take a closer look at the $(k+1)$st derivative of g. Since

  $$g^{(k+1)}(x) \sim \frac{\nu}{q^{r+1}} \alpha_0 \beta(\beta - 1) \cdots (\beta - k) x^{\beta - k - 1},$$

  we can bound it by

  $$\lambda < \frac{g^{(k+1)}(x)}{(k+1)!} < c_0 \lambda \quad (N < x \leq 2N)$$

or similarly for $-g^{(k+1)}(x)$, where

$$\lambda = c\frac{\nu}{q^{r+1}}N^{\beta-k-1}$$

and c depends only on p.

Let $\delta := \log_N c$, then for $N \to \infty$ we get that $\delta \to 0$. We may assume that N is sufficiently large such that $-1 < \delta \leq \frac{2}{3}$. Thus with help of (2.2) for r we get that

$$N^{-k-1} < N^{\delta-k} \leq \lambda \leq N^{\delta+\gamma+\beta-k-1} < N^{-1},$$

Finally an application of Lemma 2.2 yields

$$S(N,r,\nu) \ll N^{1-\eta} \tag{2.3}$$

for $\eta > 0$ and we have shown the lemma for the case of $\beta \notin \mathbb{Z}$.

- **Case 2: β is an integer.** In order to ease notation we set $b := \beta$ in this case. Since p is not a polynomial we get that there exists a $\beta_h \notin \mathbb{Z}$. Let β_h be the largest non-integer, thus, in particular $b, \beta_1, \ldots, \beta_{h-1} \in \mathbb{Z}$. Since $\beta_h \notin \mathbb{Z}$ we obviously have $h \geq 1$. Furthermore since $b \in \mathbb{Z}$ and $b > \beta_h > 1$ we have $b \geq 2$ throughout this case.

  Let ρ be such that $\gamma < \rho < \min(\beta_h - 2\gamma, 1)$. We divide the range of r up into three parts. In particular we distinguish between

$$q^{r+1} \leq N^{\beta_h-\rho}, \quad N^{\beta_h-\rho} < q^{r+1} \leq N^{b-2+\rho}, \quad N^{b-2+\rho} < q^{r+1} \leq N^{b-1}.$$

  Since we might have $b - \beta_h < 2(1-\rho)$ we note that the middle part can be empty. We will start with the lower and upper part since the treatment of the middle part is the longest one.

  ⋆ **Case 2.1: $q^{r+1} \leq N^{\beta_h-\rho}$.** As in the case of $\beta \notin \mathbb{Z}$ we want to apply Lemma 2.2. We set $g(x) := \nu q^{-(r+1)}p(x)$ and since $b, \ldots, \beta_{h-1} \in \mathbb{Z}$ we get

$$\lambda < \frac{g^{(b+1)}(x)}{(b+1)!} < c_0\lambda \quad (N < x \leq 2N)$$

  or the same for $-g^{(b+1)}(x)$, where

$$\lambda = c\frac{\nu}{q^{r+1}}N^{\beta_h-b-1}$$

  and c again only depends on p. We again set $\delta := \log_N c$ and as in **Case 1** assume that N is large enough such that $-\rho < \delta < (b-\beta_h)/2$. Since $q^{r+1} \leq N^{\beta_h-\rho}$ and $\gamma < (b-\beta_1)/2 \leq (b-\beta_h)/2$ we get that

$$N^{-b-1} < N^{\delta+\rho-b-1} \leq \lambda \leq N^{\delta+\gamma+\beta_h-b-1} < N^{-1}.$$

Thus an application of Lemma 2.2 yields

$$S(N,r,\nu) \ll N^{1-\eta}. \tag{2.4}$$

⋆ **Case 2.2:** $N^{b-2+\rho} < q^{r+1} \le N^{b-1}$**.** Since $b \ge 2$ we have

$$g''(n) > c\frac{\nu}{q^{r+1}}N^{b-2} =: \lambda > 0 \quad (N < n \le 2N).$$

We apply Lemma 2.5 in order to get

$$S(N,r,\nu) \ll \sqrt{\frac{\nu}{q^{r+1}}}N^{\frac{b}{2}} \ll N^{1-\frac{1}{2}(\rho-\gamma)}. \tag{2.5}$$

⋆ **Case 2.3:** $N^{\beta_h-\rho} < q^{r+1} \le N^{b-2+\rho}$**.** In this case we follow the proof of Theorem in [12] (*cf.* Equation (12) and the following pages), which is an adoption of the proof of Lemma 6.12 of [16]. We set

$$m := \left\lfloor \left(\frac{q^{r+1}}{\nu}\right)^{\frac{1}{b-1}} \right\rfloor \ll N^{1-\frac{1-\rho}{b-1}}. \tag{2.6}$$

Furthermore let

$$T(n) := \sum_{k=1}^{m} e(p(n+k)-p(n)).$$

Then, following Equation (6.12.4) of [16], we get

$$|S(N,r,\nu)| \le \frac{1}{m}N^{1-1/(2L)}\left(\sum_{N<n\le 2N-m}|T(n)|^{2L}\right)+m. \tag{2.7}$$

For $1 \le y \le m$ we set

$$\Delta(y) := \frac{\nu}{q^{r+1}}(p(n+y)-p(n)) - \left(t_1y+t_2y^2+\cdots+t_{b-1}y^{b-1}+\frac{\nu}{q^{r+1}}\alpha_0y^b\right).$$

Thus

$$T(n) = S(m)e(\Delta(m)) - 2\pi i\int_0^m S(y)\Delta'(y)e(\Delta(y))\mathrm{d}y, \tag{2.8}$$

where

$$\Delta'(y) = \sum_{h=1}^{b-1} h\left(\frac{\nu}{q^{r+1}}\frac{p^{(h)}(n)}{h!} - t_h\right)y^{h-1} + \frac{\nu}{q^{r+1}}\left(p^{(b)}(n+\theta y) - \alpha_0 b!\right)y^b.$$

with $0 < \theta < 1$.

For every n and $\mathbf{t} = (t_1,\ldots,t_{b-1}) \in \mathbb{R}^{b-1}$ we define $\chi(n,\mathbf{t})$ to be

$$\chi(n,\mathbf{t}) := \begin{cases} 1 & \text{if } \left\|\frac{p^{(h)}(n)}{h!} - t_h\right\| \le \frac{1}{2m^h} \quad (h = 1,2,\ldots,b-1),\\ 0 & \text{else.}\end{cases}$$

Now for fixed n, if we assume that $\chi(n,\mathbf{t}) = 1$ for $\mathbf{t} \in \mathbb{R}^{b-1}$, then we have by the definition of m in (2.6) that

$$\Delta'(y) \ll \frac{1}{m} + \frac{\nu}{q^{r+1}} N^{\beta_h - b} m^{b-1} \ll \frac{1}{m} + N^{\beta_h - b} \ll \frac{1}{m},$$

which implies for $T(n)$ that

$$T(n) \ll S(m) + \frac{1}{m}\int_0^m |S(y)|\,\mathrm{d}y.$$

Thus we get for (2.8) that

$$\sum_{N<n\le 2N-m} |T(n)|^{2L} \ll m^{\frac{1}{2}b(b-1)} \sup_{\mathbf{t}\in\mathbb{R}^{b-1}} \sum_{N<n\le 2N-m} \chi(n,\mathbf{t}) \times \int_0^1 \cdots \int_0^1 \left(|S(m)| + \frac{1}{m}\int_0^m |S(y)|\,\mathrm{d}y\right)^{2L} \mathrm{d}t_1 \cdots \mathrm{d}t_{b-1}. \tag{2.9}$$

An application of Lemma 2.4 yields for the integral

$$\begin{aligned}&\int_0^1 \cdots \int_0^1 \left(|S(m)| + \frac{1}{m}\int_0^m |S(y)|\,\mathrm{d}y\right)^{2L} \mathrm{d}t_1 \cdots \mathrm{d}t_d \\ &\quad\le 2^{2L-1} \int_0^1 \cdots \int_0^1 \left(|S(m)|^{2L} + \frac{1}{m}\int_0^m |S(y)|^{2L}\,\mathrm{d}y\right) \mathrm{d}t_1 \cdots \mathrm{d}t_{b-1} \\ &\quad\ll m^{2L - \frac{1}{2}b(b-1) + \frac{1}{2}b(b-1)(1-1/b)^R} (\log m)^R.\end{aligned} \tag{2.10}$$

Now we want to estimate the supremum by applying Lemma 2.3 with

$$\phi(n) = \frac{\nu}{q^{r+1}(b-1)!} p^{(b-1)}(n) - t_{b-1}.$$

Then $\phi(n+1) - \phi(n) = \frac{\nu}{q^{r+1}(b-1)!} p^{(b)}(n+\theta) \sim \alpha_0 b \frac{\nu}{q^{r+1}}$, so that $\delta \le \phi(n+1) - \phi(n) \le c\delta \le \frac{1}{2}$, where $\delta = \frac{1}{2}\alpha_0 b \frac{\nu}{q^{r+1}}$ and $c = 2$. If we set $W = 1/(2\delta m^{b-1})$, then an application of Lemma 2.3 gives

$$\begin{aligned}\sup_{\mathbf{t}\in\mathbb{R}^{b-1}} \sum_{N<n\le 2N-m} \chi(n,\mathbf{t}) &\ll \left(N\frac{\nu}{q^{r+1}}\alpha_0 b + 1\right)\left(\frac{1}{m^{b-1}}\frac{q^{r+1}}{\alpha_0 b\nu} + 1\right) \\ &\ll N\frac{\nu}{q^{r+1}} + 1.\end{aligned}$$

Plugging this together with (2.10) into (2.9) and then into (2.7) yields

$$S(N,r,\nu) \ll S_1(N,r,\nu) + m \tag{2.11}$$

where

$$S_1(N,r,v) = N^{1-\frac{1}{2L}}\left(m^{\frac{1}{2}b(b-1)(1-1/(b-1))^R}\left(\frac{\nu}{q^{r+1}}N+1\right)\right)^{\frac{1}{2L}}$$

with L and R any positive integers.
Now we again split the range up into two parts and consider all r such that

$$N^{\beta_h-\rho} < q^{r+1} \le N \quad \text{and} \quad N < q^{r+1} \le N^{b-2+\rho}$$

separately. We note that the lower range might be empty.
If $q^{r+1} \le N$, then clearly $\nu q^{-(r+1)}N > 1$ and we can estimate $S_1(N,r,\nu)$ by

$$\begin{aligned}
S_1(N,r,\nu) &\ll N^{1-\frac{1}{2L}}\left(q^{\frac{1}{2}(r+1)b(1-1/(b-1))^R}\frac{\nu}{q^{r+1}}N\right)^{\frac{1}{2L}}\\
&\ll Nq^{-(r+1)(1-\frac{1}{2}b(1-1/(b-1))^R)/2L}\nu^{\frac{1}{2L}}\\
&\ll N^{1-(\beta_h-\rho)(1-\frac{1}{2}b(1-1/(b-1))^R)/2L}N^{\frac{\gamma}{2L}}\\
&\ll N^{1-\frac{\beta_h-\rho}{4L}+\frac{\gamma}{2L}},
\end{aligned}$$

where we have chosen R to be sufficiently large, such that $b(1-1/(b-1))^R < 1$.
Secondly we consider the case $q^{r+1} > N$. Then we have that $\nu q^{-(r+1)}N \ll \nu$ and $S_1(N,r,\nu)$ can be estimated by

$$\begin{aligned}
S_1(N,r,\nu) &\ll N^{1-\frac{1}{2L}}m^{\frac{1}{4L}b(b-1)(1-1/(b-1))^R}\nu^{\frac{1}{2L}}\\
&\ll N^{1-\frac{1}{2L}+\frac{\gamma}{2L}}q^{\frac{r+1}{4L}b(1-1/(b-1))^R}\\
&\ll N^{1-\frac{1-\gamma}{2L}+\frac{b}{4L}(b-2+\rho)(1-1/(b-1))^R}\\
&\ll N^{1-\frac{1-\gamma}{2L}+\frac{1}{4L}}
\end{aligned}$$

where we have taken R to be sufficiently large, such that $b(b-2+\rho)(1-1/(b-1))^R < 1$.
Combining these estimates in (2.11) and using the estimate for m in (2.6) we reach at

$$S(N,r,\nu) \ll N^{1-\frac{\beta_h-\rho-2\gamma}{4L}} + N^{1-\frac{1-2\gamma}{4L}} + N^{1-\frac{1-\rho}{b-1}}. \tag{2.12}$$

Noting that $\gamma < \rho < \min(\beta_h - 2\gamma, 1)$ together with the restrictions for γ in (2.2), the estimates (2.4), (2.5) and (2.12) prove the lemma for the case $\beta \in \mathbb{Z}$.

□

Now we have to consider the case that $p(t) = \alpha_0 t^\beta$, *i.e.* $d = 0$.

Corollary 2.1. *Let N be a positive real and $p(t)=\alpha_0 t^\beta$ with $1<\beta\notin\mathbb{Z}$. If γ and r are such that*

$$0<\gamma<\{\beta\} \quad \text{and} \quad 1\leq q^{r+1}\leq N^{\beta-1} \tag{2.13}$$

hold then there exists an $\eta>0$ such that for $1\leq\nu\leq N^\gamma$ we have

$$S(N,r,\nu)=\sum_{N<n\leq 2N} e\left(\frac{\nu p(n)}{q^{r+1}}\right)\ll N^{1-\eta}.$$

Proof. Following the proof of Lemma 2.6 and in particular the case where $\beta\notin\mathbb{Z}$. Setting k and δ as there we reach at

$$N^{\delta-k}\leq\lambda\leq N^{\delta+\gamma+\beta-k-1}<N^{-1}$$

for sufficiently large N. Thus again an application of Lemma 2.2 proves the corollary. □

Now we combine our tools in order to estimate $U_{r,a}$ for r such that (1.7) holds.

Proof of Proposition 2.1. Let $\sigma>0$, which we will choose later. Then an application of Lemma 2.1 yields

$$\begin{aligned}U_{r,a}(N)&=\sum_{N<n\leq 2N}\psi\left(\frac{p(n)}{q^{r+1}}-\frac{a}{q}\right)\\&\ll N^{1-\sigma}+\sum_{\nu=1}^{N^\sigma}\frac{1}{\nu}\left|S(N,r,\nu)\right|+N^\sigma\sum_{\nu=N^\sigma+1}^{\infty}\frac{1}{\nu^2}\left|S(N,r,\nu)\right|.\end{aligned}$$

The next step is an application of Lemma 2.6 or Corollary 2.1 depending on the shape of p. Therefore we set $\gamma>0$ such that (2.2) or (2.13) hold, respectively, and get by trivially estimating $S(N,r,\nu)$ in the last term, that

$$\begin{aligned}U_{r,a}(N)&\ll N^{1-\sigma}+N^{1-(\eta-\sigma)}+N^{1-(\eta-\sigma)}\sum_{\nu=N^\sigma+1}^{N^\gamma}\frac{1}{\nu^2}+N^{1+\sigma}\sum_{\nu=N^\gamma+1}^{\infty}\frac{1}{\nu^2}\\&\ll N^{1-\sigma}+N^{1-\eta}+N^{1-(\gamma-\sigma)}.\end{aligned}$$

Now the proposition follows by setting $\sigma_1=\sigma:=\min(\eta,\gamma/2)$. □

3. Integral transform

The aim of this section is to transform the sum $U_{r,a}$ into an integral $I_{r,a}(N)$ which is defined by

$$I_{r,a}(N):=\int_{q^{-r-1}p(N)}^{q^{-r-1}p(2N)}\psi\left(t-\frac{a}{q}\right)t^{\frac{1}{\beta}-1}\mathrm{d}t. \tag{3.1}$$

In particular we will show the following.

Proposition 3.1. *Let N be positive and r be such that (1.8) holds. Then there exists an $\sigma_2 > 0$ such that*

$$U_{r,a}(N) = \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} q^{\frac{r+1}{\beta}} I_{r,a}(N) + \mathcal{O}\left(N^{1-\sigma_2}\right). \tag{3.2}$$

The main tool will be the following Lemma.

Lemma 3.1 ([9, Theorem 1.5])**.** *Let g in $[a,b]$ $(0 \le a < b)$ be a non-negative, strictly decreasing function with a continuous derivative in (a,b). If $g^{-1}(t)$ denotes the inverse function of $g(t)$, then*

$$\begin{aligned}\sum_{a<n\le b} \psi(g(n)) &- \int_a^b \psi(t)g'(t)\mathrm{d}t - \psi(a)\psi(g(a)) \\ &= \sum_{g(b)<m\le g(a)} \psi\left(g^{-1}(m)\right) - \int_{g(b)}^{g(a)} \frac{\psi(t)}{g'(g^{-1}(t))}\mathrm{d}t - \psi(b)\psi(g(b)).\end{aligned}$$

The interesting integral is the last one. For the first integral we will use partial integration and the sum is treated by the following lemma.

Lemma 3.2 ([9, Theorem 2.3])**.** *Let $g(t)$ be a real function in $[a,b]$, twice continuously differentiable. Let $g''(t)$ be monotonic and be either positive or negative throughout. Then*

$$\sum_{a<n\le b} \psi(g(n)) \ll \int_a^b \left|g''(t)\right|^{\frac{1}{3}} \mathrm{d}t + \frac{1}{\sqrt{|g''(a)|}} + \frac{1}{\sqrt{|g''(b)|}}.$$

Now we have collected all the tools for the proof of the proposition.

Proof of Proposition 3.1. In order to apply Lemma 3.1 we need that the function under consideration is monotonically decreasing. We guarantee this by rewriting the sum in (3.2) and applying Lemma 3.1, which yields

$$\begin{aligned}&\sum_{N<n\le 2N} \psi\left(\frac{p(n)}{q^{r+1}} - \frac{a}{q}\right) = \sum_{0<n\le N} \psi\left(\frac{p\left(\lfloor 2N\rfloor - n\right)}{q^{r+1}} - \frac{a}{q}\right) + \mathcal{O}(1) \\ &= -q^{-r-1}\int_0^N \psi(t)p'(\lfloor 2N\rfloor - t)\mathrm{d}t \\ &\quad + \sum_{q^{-r-1}p(\lfloor 2N\rfloor - N)-\frac{a}{q}<m\le q^{-r-1}p(\lfloor 2N\rfloor)-\frac{a}{q}} \psi\left(\lfloor 2N\rfloor - p^{-1}\left(q^{r+1}\left(m+\frac{a}{q}\right)\right)\right) \\ &\quad + q^{r+1}\int_{q^{-r-1}p(\lfloor 2N\rfloor - N)}^{q^{-r-1}p(\lfloor 2N\rfloor)} \frac{\psi\left(t-\frac{a}{q}\right)}{p'(p^{-1}(q^{r+1}t))}\mathrm{d}t + \mathcal{O}(1).\end{aligned} \tag{3.3}$$

We will write $\asymp$ if both $\ll$ and $\gg$ hold. Since p is a pseudo-polynomial we can write $p(t) = \alpha_0 t^\beta + \mathcal{O}(t^{\beta_1})$, where we set $\beta_1 = 0$ if $d = 0$ (*i.e.*,

$p(t)=\alpha_0 t^\beta$). This yields for the inverse of p

$$p^{-1}(t)=\left(\frac{t}{\alpha_0}\right)^{\frac{1}{\beta}}+\mathcal{O}\left(t^{\frac{\beta_1+1}{\beta}-1}\right). \tag{3.4}$$

Implicitly calculating the derivatives gives

$$\begin{aligned}\left(p^{-1}\right)'(t)&=\frac{1}{p'(p^{-1}(t))}=\frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}t^{\frac{1}{\beta}-1}+\mathcal{O}\left(t^{\frac{\beta_1+1}{\beta}-2}\right),\\ \left(p^{-1}\right)''(t)&=-\frac{p''(p^{-1}(t))}{(p'(p^{-1}(t)))^3}\asymp-\frac{\beta-1}{\beta^2}\alpha_0^{-\frac{1}{\beta}}t^{\frac{1}{\beta}-2}.\end{aligned} \tag{3.5}$$

Now we define $\psi_1(x):=\int_0^x\psi(t)\mathrm{d}t$ for $x\in\mathbb{R}$ and consider the three parts of (3.3) separately. It is clear, that $\psi_1(x)$ is continuous, bounded and piecewise continuously differentiable. Integration by parts for the first integral in (3.3) yields the estimate

$$\begin{aligned}\int_0^N\psi(t)p'(\lfloor 2N\rfloor-t)\mathrm{d}t&=\psi_1(t)p'(\lfloor 2N\rfloor-t)|_0^N+\int_0^N\psi_1(t)p''(\lfloor 2N\rfloor-t)\mathrm{d}t\\ &\ll N^{\beta-1}.\end{aligned} \tag{3.6}$$

In order to estimate the sum we define $g(t):=\lfloor 2N\rfloor-p^{-1}(q^{r+1}(t+\frac{a}{q}))$ for $t\geq 0$. For the range $q^{-r-1}p(\lfloor 2N\rfloor-N)-\frac{a}{q}<t\leq q^{-r-1}p(\lfloor 2N\rfloor)-\frac{a}{q}$, we have $t\asymp q^{-r-1}N^\beta$ and thus by (3.5)

$$\left|g''(t)\right|\asymp q^{(r+1)/\beta}t^{1/\beta-2}\asymp q^{2(r+1)}N^{1-2\beta}.$$

Then an application of Lemma 3.2 together with noting the bounds for q^{r+1} in this range (1.8) yield for the sum in (3.3) that

$$\begin{aligned}\sum_{q^{-r-1}p(\lfloor 2N\rfloor-N)-\frac{a}{q}<m\leq q^{-r-1}p(\lfloor 2N\rfloor)-\frac{a}{q}}\psi(g(m))&\ll q^{-(r+1)/3}N^{(\beta+1)/3}+q^{-r-1}N^{\beta-\frac{1}{2}}\\ &\ll N^{\frac{2}{3}}.\end{aligned} \tag{3.7}$$

Now we aim for the main term which we extract from the second integral of (3.3). Noting that $\psi(t) \ll 1$ we get with (3.5) that

(3.8)

$$
\begin{aligned}
&q^{r+1}\int_{q^{-r-1}p(\lfloor 2N\rfloor - N)}^{q^{-r-1}p(\lfloor 2N\rfloor)} \frac{\psi\left(t-\frac{a}{q}\right)}{p'(p^{-1}(q^{r+1}t))}\mathrm{d}t\\
&\quad = q^{r+1}\int_{q^{-r-1}p(\lfloor 2N\rfloor - N)}^{q^{-r-1}p(\lfloor 2N\rfloor)} \psi\left(t-\frac{a}{q}\right)\Bigg(\frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}(q^{r+1}t)^{\frac{1}{\beta}-1}\\
&\qquad\qquad\qquad\qquad +\mathcal{O}\left((q^{r+1}t)^{\frac{\beta_1+1}{\beta}-2}\right)\Bigg)\,\mathrm{d}t\\
&\quad = \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}q^{\frac{r+1}{\beta}}\int_{q^{-r-1}p(\lfloor 2N\rfloor - N)}^{q^{-r-1}p(\lfloor 2N\rfloor)} \psi\left(t-\frac{a}{q}\right)t^{\frac{1}{\beta}-1}\mathrm{d}t\\
&\qquad + \mathcal{O}\left(\int_{q^{-r-1}p(\lfloor 2N\rfloor - N)}^{q^{-r-1}p(\lfloor 2N\rfloor)} \psi\left(t-\frac{a}{q}\right)q^{(r+1)\left(\frac{\beta_1+1}{\beta}-1\right)}t^{\frac{\beta_1+1}{\beta}-2}\mathrm{d}t\right)\\
&\quad = \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}q^{\frac{r+1}{\beta}}\Bigg(I_{r,a}(N)\\
&\qquad\qquad + \left\{\int_{q^{-r-1}p(\lfloor 2N\rfloor - N)}^{q^{-r-1}p(N)} + \int_{q^{-r-1}p(2N)}^{q^{-r-1}p(\lfloor 2N\rfloor)}\right\}\psi\left(t-\frac{a}{q}\right)t^{\frac{1}{\beta}-1}\mathrm{d}t\Bigg)\\
&\qquad + \mathcal{O}\left(\max(N^{\beta_1+1-\beta}, \log N)\right)\\
&\quad = \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}q^{\frac{r+1}{\beta}}I_{r,a}(N) + \mathcal{O}\left(1\right) + \mathcal{O}\left(\max(N^{\beta_1+1-\beta}, \log N)\right)
\end{aligned}
$$

Plugging (3.6), (3.7) and (3.8) into (3.3) proves the proposition with $\sigma_2 = \min\left(\beta_1 + 1 - \beta, \frac{1}{3}\right)$. □

4. Proof of Theorem 1.1

In the sections above we have collected all the tools we need along the road we will follow in order to proof Theorem 1.1. We split the sum in (1.5) up into dyadic parts

$$
\sum_{n\le x} f(\lfloor p(n)\rfloor) = \sum_{1\le i\le \log_2 x} S(2^{-i}x) + \mathcal{O}(1), \tag{4.1}
$$

where

$$
S(N) := \sum_{N<n\le 2N} f(\lfloor p(n)\rfloor). \tag{4.2}
$$

The proof proceeds in three steps.

(1) In the first one, we use the q-additivity of f in order to count the occurrences of every digit separately. Therefore we will sum over all

positions and exchange the order of summation. As we will see, the resulting sums will be our well known $U_{r,a}$. Thus we will apply our estimates from Section 2 and 3.

(2) Then we will give the integral resulting from the transformation in Section 3 the shape of a periodic function such as in the results of Delange [1] and Peter [13].

(3) Finally we will put all together in order to gain the desired result.

4.1. Rewriting the sum. The idea is to use the q-additivity of f and exchange the sum over n with the sum over the digits. Therefore we need a function which indicates if the digit at position r is a or not. The main idea is to extract the main term and write the rest as sum of ψ functions, *i.e.*

(4.3)
$$\mathcal{I}_{r,a}(x)=\frac{1}{q}+\psi\left(\frac{x}{q^{r+1}}-\frac{a+1}{q}\right)-\psi\left(\frac{x}{q^{r+1}}-\frac{a}{q}\right)=\begin{cases}1 & \text{if } d_r(x)=a,\\ 0 & \text{else.}\end{cases}$$

We define $R(N)$ to be the length of the expansion of $p(N)$, *i.e.*

$$R(N):=\log_q(p(N)). \tag{4.4}$$

Since p is a pseudo-polynomial this will provide us with upper bounds for the maximum length of an expansion. Using the representations (4.3) and (1.6) together with (4.4) in (4.2) we get

(4.5)
$$\begin{aligned}S(N)&=\sum_{N<n\leq 2N}\sum_{0\leq r\leq R(2N)}\sum_{a=0}^{q-1}f(a)\mathcal{I}_{r,a}(p(n))\\&=\mu_f N\left(\lfloor R(2N)\rfloor+1\right)+\left(\sum_{a=0}^{q-1}f(a)\sum_{0\leq r\leq R(2N)}U_{r,a+1}(N)-U_{r,a}(N)\right)\\&\quad+\mathcal{O}(1).\end{aligned}$$

The terms $U_{r,a}$ occurring in (4.5) are treated in Section 2 and Section 3. Thus applying Proposition 2.1 and 3.1 and setting $\sigma=\min(\sigma_1,\sigma_2)$ we gain

$$\begin{aligned}S(N)&=\mu_f N\left(\lfloor R(2N)\rfloor+1\right)+\mathcal{O}\left(N^{1-\sigma}\right)\\&\quad+\frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}\sum_{a=0}^{q-1}f(a)\sum_{0\leq r\leq R(2N)}q^{(r+1)/\beta}\left(I_{r,a+1}(N)-I_{r,a}(N)\right)\\&\quad-\frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}}\sum_{a=0}^{q-1}f(a)\sum_{q^{r+1}\leq N^{\beta-1}}q^{(r+1)/\beta}\left(I_{r,a+1}(N)-I_{r,a}(N)\right).\end{aligned} \tag{4.6}$$

4.2. Extraction of the periodic function. The goal of this intermediate section is the transformation of the integral $I_{r,a}$ to give the result the shape as in the results of Delange [1] and Peter [13].

Since we will often use the functions $J_{f,\beta}$ and $F_{f,\beta}$ defined in (1.4), we will write $J := J_{f,\beta}$ and $F := F_{f,\beta}$ for short. Then noting the definition of J together with partial integration as used in (3.6) we get

(4.7)
$$J(x) = \sum_{a=0}^{q-1} f(a) \int_0^x \left(\psi\left(t - \frac{a+1}{q}\right) - \psi\left(t - \frac{a}{q}\right)\right) t^{\frac{1}{\beta}-1} \mathrm{d}t \ll 1 \quad (x \geq 0).$$

Now we concentrate on the first sum over the r of the $I_{r,a}$ in (4.6). By noting the definition of J in (1.4) we get that

$$\begin{aligned}
&\frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} \sum_{r\leq R(2N)} q^{(r+1)/\beta} \sum_{a=0}^{q-1} f(a)\left(I_{r,a+1}(N) - I_{r,a}(N)\right) \\
&= \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} \left(\sum_{r\leq R(2N)} q^{(r+1)/\beta} J(q^{-r-1}p(2N)) \right. \\
&\qquad \left. - \sum_{r\leq R(2N)} q^{(r+1)/\beta} J(q^{-r-1}p(N))\right) \\
&=: S_1(N) - S_2(N), \quad N \geq 1.
\end{aligned} \tag{4.8}$$

Since $p(t) = \alpha_0 t^\beta + \mathcal{O}(t^{\beta_1})$, where we may set $\beta_1 = 0$ if $p(t) = \alpha_0 t^\beta$, and $J(x) \ll 1$ by (4.7) we get on the one hand

(4.9)
$$\begin{aligned}
S_1(N) &= \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} \sum_{r\leq R(2N)} q^{(R(2N)-r+1)/\beta} J(q^{r-R(2N)-1}p(2N)) \\
&= \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} p(2N)^{\frac{1}{\beta}} q^{(1-\{R(2N)\})/\beta} \sum_{r\leq R(2N)} q^{-\frac{r}{\beta}} J(q^{\{R(2N)\}+r-1}) \\
&= \frac{1}{\beta} 2N q^{(1-\{R(2N)\})/\beta} \sum_{r\leq R(2N)} q^{-\frac{r}{\beta}} J(q^{\{R(2N)\}+r-1}) \\
&\quad + \mathcal{O}\left(N \sum_{r\leq R(2N)} q^{-\frac{r}{\beta}}\right) \\
&= \frac{1}{\beta} 2N q^{(1-\{R(2N)\})/\beta} \left(\sum_{r\geq 0} q^{-\frac{r}{\beta}} J(q^{\{R(2N)\}+r-1}) + \mathcal{O}\left(\sum_{r>R(2N)} q^{-\frac{r}{\beta}}\right)\right) \\
&\quad + \mathcal{O}(1) \\
&= 2NF(R(2N)) - \mu_f 2N(1 - \{R(2N)\}) + \mathcal{O}(1).
\end{aligned}$$

On the other hand for $S_2(N)$ we get

$$S_2(N) = S_1\left(\frac{N}{2}\right) + \frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} \sum_{R(N)<r\leq R(2N)} q^{\frac{r+1}{\beta}} J(q^{-r-1}p(N)). \tag{4.10}$$

For the second sum we have that $R(N) < r \leq R(2N)$, which implies $q^{-r-1}p(N) < \frac{1}{q}$. Thus connecting the representations in (4.3) and (1.4) we gain for $0 \leq x < \frac{1}{q}$

$$J(x) = \int_0^x \sum_{a=0}^{q-1} f(a)\left(\mathcal{I}_{r,a}(t) - \frac{1}{q}\right) t^{\frac{1}{\beta}-1}\mathrm{d}t = -\mu_f \beta x^{\frac{1}{\beta}}. \tag{4.11}$$

Now plugging (4.9) and (4.11) into (4.10) yields

$$\begin{aligned} S_2(N) &= NF(R(N)) - \mu_f N(1 - \{R(N)\}) - \mu_f N(\lfloor R(2N)\rfloor - \lfloor R(N)\rfloor) \\ &\quad + \mathcal{O}(1). \end{aligned} \tag{4.12}$$

Finally we consider the second sum over $I_{r,a}$ in (4.6). We recall the definition of $I_{r,a}(N)$ in (3.1). Then integration by parts yields for non-negative integers a and r and $N \geq 1$, that

$$I_{r,a}(N) \ll q^{\left(1-\frac{1}{\beta}\right)(r+1)} N^{1-\beta}.$$

Thus we get

$$\frac{1}{\beta}\alpha_0^{-\frac{1}{\beta}} \sum_{a=0}^{q-1} f(a) \sum_{q^{r+1}\leq N^{\beta-1}} q^{(r+1)/\beta}\left(I_{r,a+1}(N) - I_{r,a}(N)\right) \ll 1. \tag{4.13}$$

4.3. Putting all together. Plugging (4.9), (4.12) and (4.13) into (4.6) yields

$$\begin{aligned} S(N) &= \mu_f 2N\, R(2N) - \mu_f N\, R(N) + 2NF(R(2N)) - NF(R(N)) \\ &\quad + \mathcal{O}\left(N^{1-\sigma}\right). \end{aligned} \tag{4.14}$$

Plugging this into (4.1) and summing up over $N = 2^{-i}x$ for $1 \leq i \leq \log_2 x$ yields

$$\begin{aligned} \sum_{n\leq x} f(\lfloor p(n)\rfloor) &= \sum_{1\leq i\leq \log_2 x} S(2^{-i}x) + \mathcal{O}(1) \\ &= \mu_f x \log_q(p(x)) + F_{f,\beta}\left(\log_q(p(x))\right) + \mathcal{O}\left(x^{1-\varepsilon}\right) \end{aligned}$$

for an $\varepsilon > 0$, which proves the theorem.

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