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Lower bounds of discrete moments of the derivatives of the Riemann zeta-function on the critical line

par Thomas CHRIST et Justas KALPOKAS

Résumé. Nous établissons des bornes inférieures inconditionnelles pour certains moments discrets de la fonction zêta de Riemann et de ses dérivées dans la bande critique. Nous utilisons ces moments discrets pour donner des bornes inférieures inconditionnelles pour les moments continus $I_{k,l}(T) = \int_0^T |\zeta^{(l)}(\frac{1}{2}+it)|^{2k}dt$, où l est un entier positif et $k \geq 1$ un nombre rationnel. En particulier, ces bornes inférieures sont de l'ordre de grandeur attendu pour $I_{k,l}(T)$.

Abstract. We establish unconditional lower bounds for certain discrete moments of the Riemann zeta-function and its derivatives on the critical line. We use these discrete moments to give unconditional lower bounds for the continuous moments $I_{k,l}(T) = \int_0^T |\zeta^{(l)}(\frac{1}{2}+it)|^{2k}dt$, where l is a non-negative integer and $k \geq 1$ a rational number. In particular, these lower bounds are of the expected order of magnitude for $I_{k,l}(T)$.

1. Introduction and statement of the main results

In this paper, we investigate the value-distribution of the Riemann zeta-function $\zeta(s)$ on the critical line $s = \frac{1}{2} + i\mathbb{R}$. Recall the functional equation of the zeta-function,

$$\zeta(s) = \Delta(s)\zeta(1-s), \quad \text{where} \quad \Delta(s) := 2^s\pi^{s-1}\Gamma(1-s)\sin(\tfrac{\pi s}{2}). \tag{1.1}$$

It follows immediately that $\Delta(s)\Delta(1-s) = 1$; hence $\Delta(\frac{1}{2}+it)$ lies on the unit circle for real t. For a given angle $\phi \in [0, \pi)$, we denote by $t_n(\phi)$, $n \in \mathbb{N}$, the positive roots of the equation

$$e^{2i\phi} = \Delta(\tfrac{1}{2}+it)$$

in ascending order. These roots correspond to intersections of the curve $t \mapsto \zeta(\frac{1}{2}+it)$ with straight lines $e^{i\phi}\mathbb{R}$ through the origin (see Kalpokas and Steuding [10] for more details). In particular, the points $t_n(0)$ that

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are obtained by the special choice of $\phi = 0$ correspond to intersections of $t \mapsto \zeta(\frac{1}{2} + it)$ with the real axis and are called Gram points (named after Gram [5] who observed that the first of these points seperate ordinates of consecutive zeros on the critical line).

For fixed $\phi \in [0, \pi)$, the number $N_\phi(T)$ of points $t_n(\phi)$ which lie in the intverval $(0, T]$ is asymptotically given by

$$N_\phi(T) = \frac{T}{2\pi e} \log \frac{T}{2\pi e} + O(\log T).$$

For a proof we refer to Kalpokas and Steuding [10].

In the following, we investigate the growth behaviour of discrete moments

$$S_{k,l}(T, \phi) := \sum_{0 < t_n(\phi) \leqslant T} \left| \zeta^{(l)} \left(\tfrac{1}{2} + it_n(\phi) \right) \right|^{2k},$$

of the zeta-function $\zeta(s)$, resp. its derivatives $\zeta^{(l)}(s)$, on the critical line, as $T \to \infty$. Building on methods developed by Rudnick and Soundararajan [17], resp. Milinovich and Ng [12], we shall establish an unconditional lower bound for these discrete moments.

Theorem 1.1. *For any rational $k \geqslant 1$ and any non-negative integer l, uniformly for $\phi \in [0, \pi)$, as $T \to \infty$,*

$$S_{k,l}(T, \phi) \gg_{l,k} T(\log T)^{k^2 + 2kl + 1}.$$

Theorem 1.1 generalizes a result of Kalpokas, Korolev and Steuding [11], who obtained the lower bound for $S_{k,l}(T, \phi)$ in the case $l = 0$.

Under the assumption of the Riemann hypothesis, the authors proved in [1] that for non-negative integers k and l, resp. non-negative real k if $l = 0$, uniformly for $\phi \in [0, \pi)$, as $T \to \infty$,

$$S_{k,l}(T, \phi) \ll_{l,k,\varepsilon} T(\log T)^{k^2 + 2kl + 1 + \varepsilon}$$

with any fixed $\varepsilon > 0$. Thus, $T(\log T)^{k^2 + 2kl + 1}$ seems to be the true order of magnitude for the moments $S_{k,l}(T, \phi)$ as $T \to \infty$.

Essentially, the discrete moments $S_{k,l}(T, \phi)$ act, after some suitable normalization, like a Riemann sum approximating the continuous moments

$$I_{k,l}(T) := \int_1^T \left| \zeta^{(l)} \left(\tfrac{1}{2} + it \right) \right|^{2k} dt.$$

Thus, we can deduce from the estimate for the discrete moments in Theorem 1.1 the following estimate for the continuous ones.

Corollary 1.1. *For any rational $k \geqslant 1$ and any non-negative integer l, as $T \to \infty$,*

$$I_{k,l}(T) \gg_{l,k} T(\log T)^{k^2 + 2kl}.$$

For $I_{1,l}(T)$ with a non-negative integer l and $I_{2,0}$, there are classical asymptotic extensions by Hardy & Littlewood [6] and Ingham [8] which are in agreement with the estimates above. Furthermore, we must note that in the case $l = 0$, the bounds of Corollary 1.1 were proved by Heath-Brown [7] for any positive rational k and under the assumption of the Riemann hypothesis by Ramachandra [16] for any positive real k.

Milinovich [13, Theorem 3.2] showed under the assumption of the Riemann hypothesis that $I_{k,l}(T) \ll_{l,k,\varepsilon} T(\log T)^{k^2+2kl+\varepsilon}$ for non-negative integers l and k and any $\varepsilon > 0$. This and Corollary 1.1 suggest that $T(\log T)^{k^2+2kl}$ is the true order of magnitude for the moments $I_{k,l}(T)$. Especially for $l = 0$, there are many works which give evidence for this conjectured order of magnitude: e.g. Soundararajan [19], Heath-Brown [7] and Radziwill [15].

The paper is organized as follows: In the next section we provide some preliminary results. In Section 3 we prove key Proposition 3.1 which leads to Theorem 1.1. In Section 4 we prove Corollary 1.1. In Section 5 we give an alternative proof for Corollary 1.1 and in Section 6 we close with a remark.

2. Preliminaries

Recall the function $\Delta(s)$ defined by (1.1). By Striling's formula, we get

$$\Delta(\sigma+it) = \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma-it} \exp(i(t+\tfrac{\pi}{4}))(1+O(|t|^{-1})) \quad \text{for } |t| \geqslant 1 \tag{2.1}$$

uniformly for any σ from a bounded interval. Hence,

$$\frac{1}{\Delta(s)-e^{2i\phi}} = \frac{-e^{-2i\phi}}{1-e^{-2i\phi}\Delta(s)} = -e^{-2i\phi}\left(1+\sum_{k=1}^{\infty} e^{-2ki\phi}\Delta(s)^k\right) \tag{2.2}$$

holds for $\sigma > \frac{1}{2}$. Obviously, $\Delta(\frac{1}{2}+it)$ is a complex number on the unit circle for $t \in \mathbb{R}$. Moreover, note that, for t large enough, $\Delta(\sigma+it)$ lies on the unit circle only if $\sigma = \frac{1}{2}$ (see Spira [18] and Dixon & Schoenfeld [2]).

Furthermore, $\Delta'(\frac{1}{2}+it)$ is non-vanishing for sufficiently large t as follows from the asymptotic formula

$$\frac{\Delta'}{\Delta}(\sigma+it) = -\log\frac{|t|}{2\pi} + O(|t|^{-1}), \tag{2.3}$$

which holds for $|t| \geq 1$ uniformly for any σ from a bounded interval.
By (1.1), we can write

$$\Delta(\tfrac{1}{2}+it) = e^{-2i\theta(t)}, \tag{2.4}$$

where

$$\theta(t) = \operatorname{Im}\log\left(\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right) - \frac{t}{2}\log\pi, \tag{2.5}$$

is the Riemann-Siegel theta function (see Edwards [3, page 119]) which is asymptotically given by

$$\theta(t) \sim \frac{t}{2}\log\frac{t}{2\pi e} - \frac{\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + \ldots \qquad \text{for } t \geq 1. \tag{2.6}$$

The function $\theta(t)$ is differentiable and according to (2.6)

$$\theta'(t) = \tfrac{1}{2}\log\frac{t}{2\pi e} + \tfrac{1}{2} + O(t^{-2}) \tag{2.7}$$

holds for $t \geq 1$. Hence,

$$\tfrac{1}{2}\log\frac{t}{2\pi e} < \theta'(t) < \tfrac{1}{2}\log\frac{t}{2\pi e} + 1$$

for t sufficiently large. This implies that $\theta(t)$ is monotonically increasing for t large enough.

Due to (2.4), the solutions of $\Delta(\frac{1}{2} + it) - e^{i\phi} = 0$ correspond to the solutions of

$$\theta(t) \equiv \phi \mod \pi.$$

Next, we introduce certain Dirichlet polynomials

$$X(s) = \sum_{n\leqslant X}\frac{x_n}{n^s}, \quad Y(s) = \sum_{m\leqslant Y}\frac{y_m}{m^s}, \tag{2.8}$$

where $X, Y \leqslant T$. Moreover, we define the following quantities

$$\mathcal{X}_0 = \max_{n\leqslant X}|x_n|, \quad \mathcal{Y}_0 = \max_{m\leqslant Y}|y_m|, \quad \mathcal{X}_1 = \sum_{n\leqslant X}\frac{|x_n|}{n}, \quad \mathcal{Y}_1 = \sum_{m\leqslant Y}\frac{|y_m|}{m}$$

and set

$$X_1(s) = \sum_{n\leqslant X}\frac{\overline{x_n}}{n^s}, \quad Y_1(s) = \sum_{m\leqslant Y}\frac{\overline{y_m}}{m^s}.$$

We shall use a variation of Lemma 5.1 from Ng [14]. For a proof we refer to Kalpokas, Korolev and Steuding [11, Lemma 5].

Lemma 2.1. *Suppose the series $f(s) = \sum_{n=1}^{\infty}\alpha_n n^{-s}$ converges absolutely for* Re$s > 1$ *and $\sum_{n=1}^{\infty}|\alpha_n|n^{-\sigma} \ll (\sigma-1)^{-\gamma}$ for some $\gamma \geq 0$ as $\sigma \to 1+0$. Next, let $X(s)$ and $Y(s)$ be Dirichlet polynomials as defined in* (2.8)*. Then, uniformly for $a \in (1,2]$ we have*

$$\begin{aligned} J &= \frac{1}{2\pi i}\int_{a+i}^{a+iT} f(s)X(s)Y(1-s)\frac{\Delta'(s)}{\Delta(s)}ds \\ &= -\frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{\alpha_n x_m y_{mn}}{mn} + O\left(\frac{Y^a(\log T)^2\mathcal{X}_0\mathcal{Y}_0}{(a-1)^{\gamma+1}}\right), \end{aligned}$$

where the implicit constant is absolute.

We proceed with a modified version of Lemma 6 of Gonek [4].

Lemma 2.2. *Let l be a non-negative integer and $|t| \geq 1$. Then, uniformly for σ from a bounded interval,*

$$\zeta^{(l)}(1-s) = (-1)^l \sum_{k=0}^{l} \binom{l}{k} \zeta^{(k)}(s)\Delta(1-s)\left(\log\frac{t}{2\pi}\right)^{l-k} + O(t^{\frac{\sigma}{2}-1+\epsilon}).$$

Proof. First, we note that the estimates

$$\zeta^{(l)}(\sigma+it) \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\varepsilon}, & \text{if } \sigma \leq 0, \\ |t|^{\frac{1}{2}(1-\sigma)+\varepsilon}, & \text{if } 0 < \sigma \leq 1, \\ |t|^{\varepsilon}, & \text{if } \sigma > 1, \end{cases} \tag{2.9}$$

hold for any $\varepsilon > 0$ as $t \to \infty$. The estimates for the case $l = 0$ can be found in Titchmarsh [21, Chapter V]. The estimates for $l \in \mathbb{N}$ can be deduced from the case $l = 0$ by applying Cauchy's integral formula for derivatives of analytic functions to the zeta-function in a small disc centered at $s = \sigma + it$.

Next, taking the l-th derivative of both sides of the functional equation (1.1), we get according to Leibniz's rule

$$\zeta^{(l)}(1-s) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \zeta^{(k)}(s)\Delta^{(l-k)}(1-s). \tag{2.10}$$

Initially, we will show by induction that for every non-negative integer ν

$$\Delta^{(\nu)}(1-s) = \Delta(1-s)\left(-\log\frac{t}{2\pi}\right)^{\nu} + O(t^{\sigma-\frac{3}{2}}(\log t)^{\nu-1}) \tag{2.11}$$

holds uniformly for σ from a bounded interval: The case $\nu = 0$ is obviously true. Now, suppose that the assertion (2.11) is proved for $\nu = 0, ..., \mu - 1$. Differentiating the identity

$$\Delta'(1-s) = \Delta(1-s)\frac{\Delta'}{\Delta}(1-s)$$

yields that

$$\Delta^{(\mu)}(1-s) = \sum_{\nu=0}^{\mu-1} \binom{\mu-1}{\nu} \Delta^{(\nu)}(1-s)\left(\frac{\Delta'}{\Delta}\right)^{(\mu-\nu-1)}(1-s).$$

By (2.3) and Cauchy's estimate for the derivatives of analytic functions applied to a small square centered at $1 - s$, we find that

$$\left(\frac{\Delta'}{\Delta}\right)^{(\nu)}(1-s) \ll |t|^{-1}, \quad \text{for } \nu \geq 1.$$

By the estimate above, (2.1) and (2.3), we can conclude that the assertion (2.11) holds for $\nu = \mu$; and thus, inductively, for all non-negative integers ν. The assertion of the Lemma follows now immediatelly by combining (2.9), (2.10) and (2.11). □

In the next four Lemmas, we will gather several properties of the generalized κ-th divisor function (see Heath-Brown [7, Section 2]): Let κ be a positive real number. The generalized κ-th divisor function $d_\kappa : \mathbb{N} \to \mathbb{R}$ is defined by the coefficients $d_\kappa(n)$ of

$$\zeta(s)^\kappa = \sum_{n=1}^{\infty} d_\kappa(n) n^{-s}, \qquad \sigma > 1.$$

The function $d_\kappa(n)$ is multiplicative and on prime powers given by

$$d_\kappa(p^j) = \frac{\Gamma(\kappa + j)}{\Gamma(\kappa) j!}.$$

If κ is a positive integer the definition above coincides with the definition of the divisor function

$$d_\kappa(n) = \sum_{\substack{n_1,\ldots,n_\kappa \in \mathbb{N} \\ n_1 \cdots n_\kappa = n}} 1.$$

The generalized κ-th divisor function satisfies the following properties:

Lemma 2.3. *Let κ be a positive real number and n a positive integer.*

(1) *For $\kappa \geq 0$ and $n \geq 1$,we have $d_\kappa(n) \geq 0$.*
(2) *For fixed n, $d_\kappa(n)$ increases with respect to κ.*
(3) *For fixed $\kappa \geq 0$ and $\epsilon > 0$, we have $d_\kappa(n) \ll n^\epsilon$.*
(4) *If j is an integer, then*

$$d_{\kappa j}(n) = \sum_{n = n_1 n_2 \ldots n_j} d_\kappa(n_1) d_\kappa(n_2) \ldots d_\kappa(n_j).$$

For a proof, we refer to Heath-Brown [7, Lemma 1].

Lemma 2.4. *Let λ, μ be fixed positive real numbers. Then,*

$$\sum_{n \leq x} d_\lambda(n) d_\mu(n) \asymp_{\lambda,\mu} x (\log x)^{\lambda\mu - 1}$$

and, thus,

$$\sum_{n \leq x} d_\lambda(n) d_\mu(n) n^{-1} \asymp_{\lambda,\mu} (\log x)^{\lambda\mu}.$$

The first assertion of Lemma 2.4 can be established by the Selberg-Delange method (see Tenenbaum [20, Chapter II.5]) based on Perron's formula and contour integration. The second assertion then follows by Abel's summation.

Let $\varphi(m)$ denote Euler's totient function that is defined by

$$\varphi(m) = \sum_{\substack{n \leq m \\ (n,m)=1}} 1.$$

Then, we have the following.

Lemma 2.5. *Let λ, μ be fixed positive real numbers. Then,*

$$\sum_{m\le x} d_\lambda(m) d_\mu(m) \left(\frac{\varphi(m)}{m}\right)^\mu \asymp_{\lambda,\mu} x(\log x)^{\lambda\mu-1}$$

and, thus,

$$\sum_{m\le x} d_\lambda(m) d_\mu(m) \left(\frac{\varphi(m)}{m}\right)^\mu m^{-1} \asymp_{\lambda,\mu} (\log x)^{\lambda\mu}.$$

As in Lemma 2.4, the first assertion of Lemma 2.5 can be established by the Selberg-Delange method (see Tenenbaum [20, Chapter II.5]) based on Perron's formula and contour integration. The second assertion then follows by Abel's summation.

Lemma 2.6. *For any rational $k = \frac{p}{q} \ge 0$, $m \le x^{\frac{1}{2p}}$ and x sufficiently large, we have*

$$\sum_{\substack{n\le x\\ (m,n)=1}} \frac{d_k(n)}{n} \ge \left(\frac{1}{p}\frac{\phi(m)}{m}\log x\right)^k.$$

Proof. Let $k = \frac{p}{q}$ be a non-negative rational number. We consider the sum

$$W := \sum_{\substack{n\le \xi\\ (m,n)=1}} \frac{d_{\frac{1}{q}}(n)}{n}.$$

Taking the q-th power, we get

$$W^q = \sum_{\substack{n\le \xi^q\\ (m,n)=1}} \frac{d_1(n,\xi)}{n},$$

where the coefficients $d_1(n,\xi)$ are given by

$$d_1(n,\xi) = \sum_{\substack{n_1 n_2\cdots n_q = n\\ n_1,n_2,\ldots,n_q\le \xi}} d_{\frac{1}{q}}(n_1) d_{\frac{1}{q}}(n_2)\cdots d_{\frac{1}{q}}(n_q).$$

As q is an integer, we have, according to property (4) of Lemma 2.3,

$$\sum_{n_1 n_2\cdots n_q = n} d_{\frac{1}{q}}(n_1) d_{\frac{1}{q}}(n_2)\cdots d_{\frac{1}{q}}(n_q) = d_{\frac{1}{j}\cdot j}(n) = d_1(n) \equiv 1$$

for all positve integers n. Hence, we can deduce that

$$d_1(n,\xi) = d_1(n) = 1 \quad \text{if} \quad n \le \xi$$

and

$$d_1(n,\xi) \le d_1(n) = 1 \quad \text{if} \quad n > \xi.$$

Thus, we get

$$\sum_{\substack{n\le\xi\\(m,n)=1}}\frac{1}{n}\le W^q\le\sum_{\substack{n\le\xi^q\\(m,n)=1}}\frac{1}{n}\le 2q\frac{\phi(m)}{m}\log\xi.$$

Using the inequality

$$\frac{\phi(m)}{m}\log\xi\le\sum_{\substack{n\le\xi\\(m,n)=1}}\frac{1}{n}\le 2\frac{\phi(m)}{m}\log\xi,$$

which is valid for $m\le\xi^{\frac{1}{2}}$ and ξ sufficiently large and can be established by standard techniques, we get

$$\frac{\phi(m)}{m}\log\xi\le W^q\le 2q\frac{\phi(m)}{m}\log\xi$$

for $m\le\xi^{\frac{1}{2}}$. Therefore,

$$\left(\frac{\phi(m)}{m}\log\xi\right)^{\frac{1}{q}}\le W\le\left(2q\frac{\phi(m)}{m}\log\xi\right)^{\frac{1}{q}}\tag{2.12}$$

for $m\le\xi^{\frac{1}{2}}$. Taking the p-th power of W yields that

$$W^p=\sum_{\substack{n\le\xi^p\\(m,n)=1}}\frac{d_{\frac{p}{q}}(n,\xi)}{n}$$

with coefficients

$$d_{\frac{p}{q}}(n,\xi)=\sum_{\substack{n_1n_2\cdots n_p=n\\n_1,n_2,\dots,n_p\le\xi}}d_{\frac{1}{q}}(n_1)d_{\frac{1}{q}}(n_2)\cdots d_{\frac{1}{q}}(n_p).$$

By the same reasoning as above, we obtain that

$$\sum_{\substack{n\le\xi\\(m,n)=1}}\frac{d_{\frac{p}{q}}(n)}{n}\le W^p\le\sum_{\substack{n\le\xi^p\\(m,n)=1}}\frac{d_{\frac{p}{q}}(n)}{n}$$

Using the upper bound for W^p from the above inequality and the lower bound for W from (2.12), we get

$$\sum_{\substack{n\le\xi^p\\(m,n)=1}}\frac{d_k(n)}{n}=\sum_{\substack{n\le\xi^p\\(m,n)=1}}\frac{d_{\frac{p}{q}}(n)}{n}\ge W^p\ge\left(\frac{\phi(m)}{m}\log\xi\right)^{\frac{p}{q}}.$$

for $m\le\xi^{\frac{1}{2}}$. Setting $x=\xi^p$ yields the assertion of the Lemma for $m\le x^{\frac{1}{2p}}$ $\square$

Lemma 2.7. *Let l be a non-negative integer, r and k non-negative rational numbers. Then*

$$\sum_{\substack{m\leqslant x\\ mn\leqslant x}} \frac{(\log m)^l d_r(m) d_k(mn)}{mn} \gg_{l,k,r} (\log x)^{l+kr+k}.$$

Proof. Let $k=\frac{p}{q}\geq 0$ be a rational number. We consider the sum

$$W := \sum_{\substack{m\leqslant x\\ mn\leqslant x}} \frac{(\log m)^l d_r(m) d_k(mn)}{mn} = \sum_{m\leq x} \frac{(\log m)^l d_r(m)}{m} \sum_{n\leqslant \frac{x}{m}} \frac{d_k(mn)}{n}.$$

Certainly, the following estimates hold

$$\begin{aligned} W &\geq \sum_{m\leq x} \frac{(\log m)^l d_r(m) d_k(m)}{m} \sum_{\substack{n\leqslant \frac{x}{m}\\ (m,n)=1}} \frac{d_k(n)}{n} \\ &\geq \sum_{x^{\frac{1}{3p+1}}\leq m\leq x^{\frac{1}{2p+1}}} \frac{(\log m)^l d_r(m) d_k(m)}{m} \sum_{\substack{n\leqslant x^{\frac{2p}{2p+1}}\\ (m,n)=1}} \frac{d_k(n)}{n}. \end{aligned}$$

Now, Lemma 2.6 yields

$$W \geq (3p+1)^{-l}\left(p+\tfrac{1}{2}\right)^{-k} (\log x)^{l+k} \sum_{x^{\frac{1}{3p+1}}\leq m\leq x^{\frac{1}{2+1}}} \frac{d_r(m) d_k(m)}{m}\left(\frac{\phi(m)}{m}\right)^k.$$

By Lemma 2.5, we get

$$W \gg_{k,l,r} (\log x)^{l+kr+k}$$

and the Lemma is proved. □

3. Proof of Theorem 1.1

In order to prove Theorem 1.1 we consider the discrete moments

(3.1)
$$S_1(T,\phi) = \sum_{0<t_n(\phi)\leqslant T} \zeta^{(l)}\left(\tfrac{1}{2}-it_n(\phi)\right) X\left(\tfrac{1}{2}+it_n(\phi)\right) Y\left(\tfrac{1}{2}-it_n(\phi)\right)$$

and

$$S_2(T,\phi) = \sum_{0<t_n(\phi)\leqslant T} \left|X\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^2. \tag{3.2}$$

with Dirichlet polynomials $X(s)$ and $Y(s)$ defined in (2.8). Our first aim is to prove the following.

Proposition 3.1. *Let $X(s)$ and $Y(s)$ be Dirichlet polynomials as defined in (2.8). Then uniformly for $\phi \in [0,\pi)$, as $T \to \infty$,*

(3.3)
$$S_1(T,\phi) = e^{-2i\phi} \sum_{k=0}^{l} (-1)^{l+k} \binom{l}{k} \sum_{\substack{m \leqslant X \\ mn \leqslant Y}} \frac{(\log n)^k x_m y_{mn}}{mn} \frac{T}{2\pi} P_{l-k+1}\left(\log \frac{T}{2\pi}\right) + (-1)^l \frac{T}{2\pi}\left(\log \frac{T}{2\pi e}\right) \sum_{\substack{m \leqslant Y \\ mn \leqslant X}} \frac{(\log m)^l y_m x_{mn}}{mn} + O(R_1),$$

where $P_n(x)$ is a polynomial of degree n and

$$R_1 = (X+Y)(T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + T^{\epsilon}\mathcal{X}_0\mathcal{Y}_0) + X^{\frac{1}{2}}Y^{\frac{1}{2}}T^{\frac{1}{6}+\epsilon}\mathcal{X}_0\mathcal{Y}_0 + T^{\epsilon}\mathcal{X}_0\mathcal{Y}_1;$$

moreover, uniformly for $\phi \in [0,\pi)$ as $T \to \infty$,

$$S_2(T,\phi) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi e}\right) \sum_{n \leqslant X} \frac{|x_n|^2}{n} + O(R_2), \tag{3.4}$$

where

$$R_2 = X\sqrt{T}(\log T)^2 \sum_{n \leqslant X} \frac{|x_n|^2}{n} + X(\log T)^3 \mathcal{X}_0^2.$$

Proof of Proposition 3.1. A proof of statement (3.4) can be found in [11, Proposition 9, equation (10)]. Thus, it only remains to prove (3.3).

We begin with the estimates

$$\begin{aligned} &\left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right| \ll t^{1/6+\varepsilon}, \\ &\left|X\left(\tfrac{1}{2}+it\right)\right| \leq \sum_{n \leqslant X} \frac{|x_n|}{\sqrt{n}} = \sum_{n \leqslant X} \sqrt{n}\frac{|x_n|}{n} \leq \sqrt{X}\mathcal{X}_1, \\ &\left|Y\left(\tfrac{1}{2}+it\right)\right| \leq \sqrt{Y}\mathcal{Y}_1; \end{aligned} \tag{3.5}$$

the first one follows from a well-known bound for the zeta-function on the critical line (see Titchmarsh [21, Chapter V]) in combination with Cauchy's integral formula for the derivatives of an analytic function applied to a small disc centered at s, whereas the second and third assertion are straightforward. Hence, in order to prove (3.3), it is sufficient to consider the sum over $c < t_n(\phi) \leqslant T$, where $c > 32\pi$ is a large absolute constant.

Next, without loss of generality, we can assume that $T = \frac{1}{2}(t_\nu(\phi) + t_{\nu+1}(\phi))$ for some $\nu \in \mathbb{N}$ since, by (3.5), for any $T_0 > T$ with $T_0 - T \ll 1$ we have

$$\sum_{T < t_n(\phi) \leq T_0} \zeta^{(l)}\left(\tfrac{1}{2} - it_n(\phi)\right) X\left(\tfrac{1}{2} + it_n(\phi)\right) Y\left(\tfrac{1}{2} - it_n(\phi)\right) \ll X^{\frac{1}{2}}Y^{\frac{1}{2}}T^{\frac{1}{6}+\epsilon}\mathcal{X}_0\mathcal{Y}_0.$$

Since the points $s = \frac{1}{2} + it_n(\phi)$ are the roots of the function $\Delta(s) - e^{2i\phi}$, the sum in question can be rewritten as a contour integral:

$$\sum_{c<t_n(\phi)\leqslant T} \zeta^{(l)}(\tfrac{1}{2} - it_n(\phi))X(\tfrac{1}{2} + it_n(\phi))Y(\tfrac{1}{2} - it_n(\phi))$$
$$= \frac{1}{2\pi i}\int_{\square} \zeta^{(l)}(1-s)X(s)Y(1-s)\frac{\Delta'(s)}{\Delta(s) - e^{2i\phi}}ds;$$

here $\square$ stands for the counterclockwise oriented rectangular contour with vertices $a+ic,\ a+iT,\ 1-a+iT,\ 1-a+ic$, where $a = 1 + (\log T)^{-1}$. Let $\mathcal{I}_1$ and $\mathcal{I}_3$ be integrals over right and left sides of contour, and $\mathcal{I}_2$ and $\mathcal{I}_4$ be the integrals over the top and bottom edges of the contour. We may assume the constant c so large that the relations

$$|\Delta(a+it)| = \left(\frac{t}{2\pi}\right)^{1/2-a}(1 + O(t^{-1})) \le 2\left(\frac{t}{2\pi}\right)^{-1/2} < \frac{1}{2}$$

hold for any $t > c$.

Moreover, we observe that for $s = a + it$ we have

$$|X(a+it)| \le \sum_{n\leqslant X}\frac{|x_n|}{n^a} \leqslant \mathcal{X}_1,$$
$$(3.6)\qquad |Y(1-a-it)| \le \sum_{m\leqslant Y}\frac{m^a|y_m|}{m} \ll Y\mathcal{Y}_1,$$
$$\left|\sum_{k=1}^{\infty} e^{-2ik\phi}\Delta(a+it)^k\right| \le 2\left(\frac{t}{2\pi}\right)^{-1/2}\sum_{k=0}^{\infty}\frac{1}{2^k} \ll t^{-1/2}.$$

In view of (2.2) and Lemma 2.2 we have

$$\mathcal{I}_1 = e^{-2i\phi}(-1)^{l+1}\sum_{k=0}^{l}\binom{l}{k}\int_c^T\left(\log\frac{\tau}{2\pi}\right)^{l-k}$$
$$\times\, d\left(\frac{1}{2\pi}\int_{a+ic}^{a+i\tau}\zeta^{(k)}(s)X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)\left(1 + \sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k\right)ds\right)$$
$$+\, O(YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1),$$

where the error term comes from the application of (2.2), (3.6), and the error term of Lemma 2.2, i.e.

$$\frac{1}{2\pi}\int_{a+ic}^{a+iT} O(t^{-\frac{1}{2}+\epsilon})X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)\left(1 + \sum_{k=1}^{\infty}e^{-2ik\phi}\Delta(s)^k\right)ds$$
$$\ll YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

In order to evaluate $\mathcal{I}_1$, we first consider $j_1 + j_2$ where

$$j_1 = \frac{1}{2\pi i}\int_{a+ic}^{a+i\tau} \zeta^{(k)}(s)X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)ds$$

and

$$j_2 = \frac{1}{2\pi i}\int_{a+ic}^{a+i\tau} \zeta^{(k)}(s)X(s)Y(1-s)\frac{\Delta'}{\Delta}(s)\sum_{k=1}^{\infty} e^{-2ik\phi}\Delta(s)^k ds.$$

By (3.6) we have

$$|j_2| \ll \zeta^{(k)}(a)Y\mathcal{X}_1\mathcal{Y}_1\int_c^{\tau}\frac{\log t dt}{\sqrt{t}} \ll Y\tau^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

Applying Lemma 2.1 to j_1, we get

$$j_1 = (-1)^{k+1}\frac{\tau}{2\pi}\left(\log\frac{\tau}{2\pi e}\right)\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k x_m y_{mn}}{mn} + O\left(Y\tau^{\epsilon}\mathcal{X}_0\mathcal{Y}_0\right).$$

Hence,

$$\begin{aligned}\mathcal{I}_1 =& e^{-2i\phi}\sum_{k=0}^{l}(-1)^{l+k}\binom{l}{k}\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k x_m y_{mn}}{mn}\frac{T}{2\pi}P_{l-k+1}\left(\log\frac{T}{2\pi}\right)\\ &+ O(YT^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + YT^{\epsilon}\mathcal{X}_0\mathcal{Y}_0 + T^{\epsilon}\mathcal{X}_1\mathcal{Y}_0),\end{aligned}$$

where

$$\frac{T}{2\pi}P_{l-k+1}\left(\log\frac{T}{2\pi}\right) + O(1) = \int_c^T\left(\log\frac{\tau}{2\pi}\right)^{l-k} d\left(\frac{\tau}{2\pi}\left(\log\frac{\tau}{2\pi e}\right)\right)$$

and $P_n(x)$ is a polynomial of degree n. The additional error term for $\mathcal{I}_1$ comes from the bound

$$\left| e^{-2i\phi}\sum_{k=0}^{l}(-1)^{l+k}\binom{l}{k}\sum_{\substack{m\leqslant X\\ mn\leqslant Y}}\frac{(\log n)^k x_m y_{mn}}{mn}\right| \ll T^{\epsilon}\mathcal{X}_1\mathcal{Y}_0.$$

In a similar way we may compute $\mathcal{I}_3$. We observe that

$$\mathcal{I}_3 = -\frac{1}{2\pi}\int_c^T \zeta^{(l)}(a-it)X(1-a+it)Y(a-it)\frac{\Delta'(1-a+it)}{\Delta(1-a+it)-e^{2i\phi}}dt.$$

This yields in combination with $\overline{X}(s) = X_1(\overline{s}), \overline{Y}(s) = Y_1(\overline{s})$ (see (2.8))

$$\overline{\mathcal{I}_3} = -\frac{1}{2\pi i}\int_{a+ic}^{a+iT}\zeta^{(l)}(s)X_1(1-s)Y_1(s)\frac{\Delta'(1-s)}{\Delta(1-s)-e^{-2i\phi}}ds.$$

In view of (2.2) we find that

$$\overline{\mathcal{I}_3} = -\frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta^{(l)}(s) X_1(1-s) Y_1(s) \frac{\Delta'}{\Delta}(s)\Big(1+\sum_{k=1}^{\infty} e^{-2ik\phi}\Delta(s)^k\Big) ds$$
$$= -(j_3 + j_4),$$

where

$$j_3 = \frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta^{(l)}(s) X_1(1-s) Y_1(s) \frac{\Delta'}{\Delta}(s) ds$$

and

$$j_4 = \frac{1}{2\pi i}\int_{a+ic}^{a+iT} \zeta^{(l)}(s) X_1(1-s) Y_1(s) \frac{\Delta'}{\Delta}(s) \sum_{k=1}^{\infty} e^{-2ik\phi}\Delta(s)^k ds.$$

By (3.6) we get

$$|j_4| \ll X T^{\frac{1}{2}+\epsilon} \mathcal{X}_1 \mathcal{Y}_1.$$

Using Lemma 2.1, we find that

$$\overline{\mathcal{I}_3} = (-1)^l \frac{T}{2\pi}\Big(\log\frac{T}{2\pi e}\Big) \sum_{\substack{m\leqslant Y\\ mn\leqslant X}} \frac{(\log m)^l\, \overline{y_m}\, \overline{x_{mn}}}{mn}$$
$$+ O(X T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 + X T^{\epsilon}\mathcal{X}_0\mathcal{Y}_0).$$

In order to estimate $\mathcal{I}_2$ we first note that the following inequalities hold along the line segment of the integration:

$$|\zeta^{(l)}(1-s)| \ll T^{\frac{1}{2}+\epsilon}, \quad |X(s)| \le \sum_{n\leqslant X}\frac{|x_n|}{n} n^{1-\sigma} \ll X^{1-\sigma}\mathcal{X}_1,$$
$$|Y(1-s)| \le \sum_{n\leqslant Y}\frac{|y_n|}{n} n^{\sigma} \ll Y^{\sigma}\mathcal{Y}_1,$$

and, finally,

$$|\zeta^{(l)}(1-s) X(s) Y(1-s)| \ll T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1 X\Big(\frac{Y}{X}\Big)^{\sigma}$$
$$\ll X T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1\Big\{\Big(\frac{Y}{X}\Big)^{1-a} + \Big(\frac{Y}{X}\Big)^{a}\Big\}$$
$$\ll (X+Y) T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

Next, by (2.3) we get

$$\frac{\Delta'(s)}{\Delta(s)-e^{2i\phi}} = \frac{\Delta'(s)}{\Delta(s)}\Big(1+\frac{e^{2i\phi}}{\Delta(s)-e^{2i\phi}}\Big) \ll (\log T)\Big(1+\frac{1}{|\Delta(s)-e^{2i\phi}|}\Big).$$

The second term in the brackets is bounded by an absolute constant. Indeed, in the case $\sigma \geqslant \frac{1}{2}+\frac{1}{3}\big(\log\frac{T}{2\pi}\big)^{-1}$ we have by (2.1) for sufficiently large T

$$|\Delta(\sigma+iT)| = \left(\frac{T}{2\pi}\right)^{1/2-\sigma}(1+O(T^{-1})) \le e^{-1/3}(1+O(T^{-1})) < \frac{1}{2},$$

and hence $|\Delta(s)-e^{2i\phi}| \geqslant 1-|\Delta(s)| > \frac{1}{2}$. Similarly, in the case $\sigma \leqslant \frac{1}{2}-\frac{1}{3}\big(\log\frac{T}{2\pi}\big)^{-1}$ we get for sufficiently large T

$$|\Delta(\sigma+iT)| \ge e^{1/3}(1+O(T^{-1})) > \frac{4}{3}, \quad |\Delta(s)-e^{2i\phi}| > \frac{4}{3}-1 = \frac{1}{3}.$$

Finally, let

$$\frac{1}{2}-\frac{1}{3}\left(\log\frac{T}{2\pi}\right)^{-1} < \sigma < \frac{1}{2}+\frac{1}{3}\left(\log\frac{T}{2\pi}\right)^{-1}.$$

Then, using the relations

$$\Delta(\tfrac{1}{2}+iT) = e^{-2i\vartheta(T)}, \quad \Delta(\sigma+iT) = \tau e^{-2i\vartheta(T)}(1+O(T^{-1})),$$

where $\tau = \big(T/(2\pi)\big)^{1/2-\sigma}$ and $\vartheta=\vartheta(T)$ denotes the increment of any fixed continuous branch of the argument of $\pi^{-s/2}\Gamma(s/2)$ along the line segment with end-points $s=\frac{1}{2}$ and $s=\frac{1}{2}+iT$, we have $e^{-1/3}\leqslant\tau\leqslant e^{1/3}$ and

$$\begin{aligned}\Delta(\sigma+iT)-e^{2i\phi} &= \big(\Delta(\sigma+iT)-\Delta(\tfrac{1}{2}+iT)\big)+\big(\Delta(\tfrac{1}{2}+iT)-e^{2i\phi}\big)\\ &= (\tau-1)e^{-2i\vartheta}-2ie^{i(\phi-\vartheta)}\sin(\phi+\vartheta)+O(T^{-1})\\ &= e^{-i\vartheta}\big((\tau-1)\cos\vartheta+2\sin(\vartheta+\phi)\sin\phi-\\ &\quad - i((\tau-1)\sin\vartheta+2\sin(\vartheta+\phi)\cos\phi)\big)+O(T^{-1}).\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\big|\Delta(\sigma+iT)-e^{2i\phi}\big|^2 &= (\tau-1)^2+4\tau\sin^2(\vartheta+\phi)+O(T^{-1})\\ &\geqslant 4\tau\sin^2(\vartheta+\phi)+O(T^{-1}).\end{aligned}$$

Using the fact that $T=\frac{1}{2}(t_\nu(\phi)+t_{\nu+1}(\phi))$ for some ν, we finally get

$$\sin^2(\vartheta+\phi) = \sin^2\left(\pi\nu+\frac{\pi}{2}+O(T^{-1})\right) \ge \sin^2\frac{\pi}{3} = \frac{3}{4}$$

for sufficiently large T, and hence,

$$|\Delta(\sigma+iT)-e^{2i\phi}|^2 \ge 4\cdot\frac{3}{4}e^{-1/3}+O(T^{-1}) > 2.$$

Thus, $|\Delta(s)-e^{2i\phi}| > \frac{1}{3}$ for any s under consideration. Hence

$$\mathcal{I}_2 \ll (X+Y)T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1.$$

The integral $\mathcal{I}_4$ can be estimated in a similar way and, thus, relation (3.3) is proved. □

Now we can proceed to proof Theorem 1.1.

Proof of Theorem 1.1. Suppose that $k = \frac{p}{q}$ is a rational number with $p > q \geqslant 1$ and $(p,q) = 1$. Let l be a non-negative integer. We set $r := p - q$ and choose $\xi := T^{1/(4p)}$. First, we define fixed coefficients for the Dirichlet polynomials $X(s)$ and $Y(s)$ in (2.8) via

$$X(s) = \Big(\sum_{n\leqslant\xi} \frac{d_{\frac{1}{q}}(n)}{n^s}\Big)^p = \sum_{n\leqslant\xi^p} \frac{d_{\frac{p}{q}}(n;\xi)}{n^s},$$

$$Y(s) = \Big(\sum_{n\leqslant\xi} \frac{d_{\frac{1}{q}}(n)}{n^s}\Big)^r = \sum_{n\leqslant\xi^r} \frac{d_{\frac{r}{q}}(n;\xi)}{n^s},$$

where $d_{\frac{m}{q}}(n;\xi)$ is given by

$$d_{\frac{m}{q}}(n;\xi) = \sum_{\substack{n=n_1\cdots n_m \\ n_1,\dots,n_m\leqslant\xi}} d_{\frac{1}{q}}(n_1)\dots d_{\frac{1}{q}}(n_m)$$

for $m = p, r$. From property (5) of Lemma 2.3 we can easily deduce that $d_{\frac{m}{q}}(n;\xi) = d_{\frac{m}{q}}(n)$ for $m \leqslant \xi$ and $0 \leqslant d_{\frac{m}{q}}(n,\ \xi) \leqslant d_{\frac{m}{q}}(n)$ for $m > \xi$.

Now, let $S_1(T,\varphi)$ and $S_2(T,\varphi)$ be the moments given by (3.1), resp. (3.2), with respect to the above chosen Dirichlet polynomials $X(s)$ and $Y(s)$. Hölder's inequality assures that

$$\begin{aligned} |S_1(T,\phi)| \leq & \Big(\sum_{0<t_n(\phi)\leqslant T} \big|\zeta^{(l)}\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k}\Big)^{1/(2k)} \times \\ & \times \Big(\sum_{0<t_n(\phi)\leqslant T} \big|X\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k/(2k-1)} \cdot \\ & \qquad \cdot \big|Y\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k/(2k-1)}\Big)^{1-1/(2k)} \\ & = \Big(\sum_{0<t_n(\phi)\leqslant T} \big|\zeta^{(l)}\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k}\Big)^{1/(2k)} \big(S_2(T,\phi)\big)^{1-1/(2k)}. \end{aligned}$$

Thus, we have

$$\sum_{0<t_n(\phi)\leqslant T} \big|\zeta\big(\tfrac{1}{2}+it_n(\phi)\big)\big|^{2k} \geq \frac{\big(S_1(T,\phi)\big)^{2k}}{\big(S_2(T,\phi)\big)^{2k-1}}.$$

We proceed with bounding $S_1(T,\varphi)$ from below: by statement (3.3) in Proposition 3.1, we have

$$S_1(T,\phi) = \sum_{j=0}^{l}(-1)^{l+j}\binom{l}{j}\frac{T}{2\pi}P_{l-j+1}\left(\log\frac{T}{2\pi}\right)\Sigma_1 + \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\Sigma_2 + O(R_1).$$

By Lemma 2.4,

$$\begin{aligned}\Sigma_1 &= \sum_{m\leqslant \xi^p, mn\leqslant \xi^r}\frac{(\log n)^j d_{\frac{p}{q}}(m;\xi)d_{\frac{r}{q}}(mn;\xi)}{mn}\\ &\leq (\log \xi^r)^j\sum_{n\leq \xi^r}\frac{d_{\frac{r}{q}}(n)}{n}\sum_{l\mid n}d_{\frac{p}{q}}(l)\\ &= (\log \xi^r)^j\sum_{n\leqslant \xi^r}\frac{d_{\frac{r}{q}}(n)d_{\frac{p}{q}+1}(n)}{n}\\ &\ll (\log T)^{(\frac{p}{q})^2-1+j},\end{aligned}$$

and by Lemma 2.7,

$$\begin{aligned}\Sigma_2 &= \sum_{m\leqslant \xi^r, mn\leqslant \xi^p}\frac{(\log m)^l d_{\frac{r}{q}}(m;\xi)d_{\frac{p}{q}}(mn;\xi)}{mn}\\ &\geq \sum_{m\leqslant \xi}\frac{(\log m)^l d_{\frac{r}{q}}(m)d_{\frac{p}{q}}(mn)}{mn}\\ &\gg (\log \xi)^{(\frac{p}{q})^2+l}.\end{aligned}$$

The error term of $S_1(T,\phi)$ is bounded by

$$\begin{aligned}R_1 &\ll (\xi^p+\xi^r)T^{\frac{1}{2}+\epsilon}\sum_{n\leqslant \xi^p}\frac{d_{\frac{p}{q}}(n;\xi)}{n}\sum_{m\leqslant \xi^r}\frac{d_{\frac{r}{q}}(m;\xi)}{m} + \xi^p\xi^r T^{\frac{1}{6}+\epsilon}\\ &\ll T^{3/4+\epsilon} \ll T^{4/5}.\end{aligned}$$

Thus, we obtain that

$$|S_1(T,\phi)| \gg T(\log T)^{k^2+l+1}.$$

Moreover, for $S_2(T,\phi)$ we have by statement (3.4) of Proposition 3.1 and Corollary 2.4 that

$$|S_2(T,\phi)| = \frac{T}{2\pi}\left(\log\frac{T}{2\pi e}\right)\sum_{n\leqslant\xi^p}\frac{d^2_{\frac{p}{q}}(n;\xi)}{n} + O(\xi^p\sqrt{T}(\log T)^{k^2+1})$$
$$\ll T(\log T)^{k^2+1}.$$

Altogether, we get

$$\sum_{0<t_n(\phi)\leqslant T}\left|\zeta\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k} \geq \frac{\left(S_1(T,\phi)\right)^{2k}}{\left(S_2(T,\phi)\right)^{2k-1}} \gg T(\log T)^{k^2+2kl+1}$$

and Theorem 1.1 is proved. □

4. Proof of Corollary 1.1

In order to prove Corollary 1.1 we will use the following proposition which allows us to express the continuous moments $I_{k,l}(T)$ in terms of the discrete moments $S_{k,l}(T,\phi)$.

Proposition 4.1. *Let k be any non-negative real number and l any non-negative integer. Then, for T large enough,*

$$\int_T^{2T}\left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k}\theta'(t)dt = \int_0^{\pi}\sum_{T\leq t_n(\phi)\leq 2T}\left|\zeta^{(l)}\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k}d\phi.$$

Proof. We set $g(t) := \left|\zeta^{(l)}(\tfrac{1}{2}+it)\right|^{2k}$ and choose a constant $c>0$ such that the Riemann-Siegel theta function $\theta(t)$ is monotonically increasing for $t>c$. Let $T>c$ and $T_i := t_{M+i}(0)$ with $i=0,...,N$ denote the Gram points that lie in the interval $[T,2T]$. We define a smooth function $[c/\pi,\infty)\ni x\mapsto t_x$ via

$$\theta(t_x) = \pi\cdot x.$$

Then, $t_{n+\phi/\pi} = t_n(\phi)$ for every positive integer n and every $\phi\in[0,\pi)$. Hence, we get

$$\begin{aligned}(4.1)\qquad \int_{t_n}^{t_{n+1}} g(t)\theta'(t)dt &= \int_{t_n}^{t_{n+1}} g(t)d\theta(t) = \int_0^1 g(t_{n+u})d\theta(t_{n+u})\\ &= \int_0^1 g(t_{n+u})d(\pi(n+u)) = \int_0^1 g(t_{n+u})\pi du\\ &= \int_0^{\pi} g(t_{n+\phi/\pi})\pi d(\phi/\pi) = \int_0^{\pi} g(t_n(\phi))d\phi.\end{aligned}$$

Therefore,

$$\int_{T_1}^{T_N} g(t)\theta'(t)dt = \int_{T_1}^{T_N} g(t)d\theta(t) = \sum_{M\leq n\leq M+N}\int_{t_n}^{t_{n+1}} g(t)d\theta(t) =$$

$$= \int_0^\pi \left(\sum_{M \le n \le M+N} g(t_n(\phi)) \right) d\phi = \int_0^\pi \left(\sum_{T_1 \le t_n(\phi) \le T_N} g(t_n(\phi)) \right) d\phi.$$

Noting that the segments $[T, T_1]$ and $[T_N, 2T]$ can be treated in a way analogue to (4.1), the assertion of the Proposition follows. □

We are now ready to prove Corollary 1.1.

Proof of Corollary 1.1. Using the asymptotic extension (2.7) for $\theta'(t)$, Proposition 4.1 yields for any rational $k \ge 1$ and any non-negative integer l

$$\int_T^{2T} \left| \zeta^{(l)} \left(\tfrac{1}{2} + it \right) \right|^{2k} dt \asymp \frac{1}{\log T} \int_0^\pi \sum_{T \le t_n(\phi) \le 2T} \left| \zeta^{(l)} \left(\tfrac{1}{2} + it_n(\phi) \right) \right|^{2k} d\phi.$$

Combining this with Theorem 1.1, we get for any rational $k \ge 1$ and any non-negative integer l

$$\begin{aligned} \int_1^T |\zeta^{(l)}(\tfrac{1}{2} + it)|^{2k} dt &\ge \sum_{j=0}^\infty \int_{T/2^{j+1}}^{T/2^j} |\zeta^{(l)}(\tfrac{1}{2} + it)|^{2k} dt \\ &\gg \sum_{j=0}^\infty \frac{1}{\log T} \int_0^\pi \sum_{\frac{T}{2^{j+1}} \le t_n(\phi) \le \frac{T}{2^j}} |\zeta^{(l)}(\tfrac{1}{2} + it_n(\phi))|^{2k} d\phi \\ &\gg T(\log T)^{k^2 + 2kl}. \end{aligned}$$

Thus, Corollary 1.1 follows. □

5. Alternative proof of Corollary 1.1

Using a method of Rudnick and Soundararajan [17], the assertion of Corollary 1.1 can be proved in a direct way without relying on the discrete moments $S_{k,l}(T, \phi)$ and Theorem 1.1. We will demonstrate this proof for the case $l = 0$:

By Cauchy's residue theorem we have

$$\frac{1}{2\pi i} \left(\int_{a+i}^{a+iT} + \int_{a+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}+i} + \int_{\frac{1}{2}+i}^{a+i} \right) \zeta(s) X(s) Y(1-s) ds = 0,$$

where $X(s)$ and $Y(s)$ are defined by (2.8) and $a = 1 + (\log T)^{-1}$. We can conclude that

$$\frac{1}{2\pi i} \int_{\frac{1}{2}+i}^{\frac{1}{2}+iT} \zeta(s) X(s) Y(1-s) ds$$

$$= \frac{1}{2\pi i} \left(\int_{a+i}^{a+iT} + \int_{a+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+i}^{a+i} \right) \zeta(s) X(s) Y(1-s) ds.$$

The second and the third integral on the right hand side are bounded by $\ll (X+Y)T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1$, (see the proof of (3.3)). Thus, we get

$$\int_1^T \zeta(\tfrac{1}{2}+it)X(\tfrac{1}{2}+it)Y(\tfrac{1}{2}-it)dt$$
$$= \frac{1}{i}\int_{a+i}^{a+iT} \zeta(s)X(s)Y(1-s)ds + O((X+Y)T^{\frac{1}{2}+\epsilon}\mathcal{X}_1\mathcal{Y}_1).$$

The integral on the right hand side can be evaluated as

$$\sum_{n=1}^{\infty}\frac{1}{n^a}\sum_{m\le X}\frac{x_m}{m^a}\sum_{k\le Y}\frac{y_k}{k^{1-a}}\frac{1}{2\pi}\int_1^T\left(\frac{k}{mn}\right)^{it}dt$$
$$= T\sum_{\substack{m\le X\\ mn\le Y}}\frac{x_m y_{mn}}{mn} + O(Y(\log T)^4\mathcal{X}_0\mathcal{Y}_0),$$

where the error term comes from the off-diagonal terms (see the proof of Lemma 5 in Kalpokas, Korolev and Steuding [11]). In a similar way we can show that

$$\int_1^T |X(\tfrac{1}{2}+it)|^2dt = T\sum_{n\leqslant X}\frac{|x_n|^2}{n} + O(R_2),$$

where R_2 is the same as in (3.4).

Now, we follow the proof of Theorem 1.1, where k, p, q, r, X, Y, ξ are the same. We set

$$A(s) := \sum_{n\leqslant\xi}\frac{d_{\frac{1}{q}}(n)}{n^s},$$
$$X(s) := \left(\sum_{n\leqslant\xi}\frac{d_{\frac{1}{q}}(n)}{n^s}\right)^r = \sum_{n\leqslant\xi^r}\frac{d_{\frac{r}{q}}(n;\xi)}{n^s},$$
$$Y(s) := \left(\sum_{n\leqslant\xi}\frac{d_{\frac{1}{q}}(n)}{n^s}\right)^p = \sum_{n\leqslant\xi^p}\frac{d_{\frac{p}{q}}(n;\xi)}{n^s}.$$

Hence,

$$A(s)^{k-1} = X(s) \quad \text{and} \quad A(s)^k = Y(s).$$

By Hölder's inequality we get

$$\left|\int_1^T \zeta(\tfrac{1}{2}+it)A(\tfrac{1}{2}+it)^{k-1}A(\tfrac{1}{2}-it)^k dt\right|$$
$$\le \left(\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k}dt\right)^{\frac{1}{2k}}\left(\int_1^T |A(\tfrac{1}{2}+it)^{k-1}A(\tfrac{1}{2}-it)^k|^{\frac{2k}{2k-1}}dt\right)^{\frac{2k-1}{2k}}.$$

Thus, we have

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \geq \frac{\left|\int_1^T \zeta(\frac{1}{2}+it)A(\frac{1}{2}+it)^{k-1}A(\frac{1}{2}-it)^k dt\right|^{2k}}{\left(\int_1^T |A(\frac{1}{2}+it)|^{2k} dt\right)^{2k-1}} =: \frac{|S_1|^{2k}}{S_2^{2k-1}}.$$

First, we bound $|S_1|$ from below. We have

$$|S_1| \gg T \sum_{\substack{m \leq X \\ mn \leq Y}} \frac{d_{\frac{r}{q}}(m;\xi) d_{\frac{p}{q}}(mn;\xi)}{mn} \gg T(\log T)^{(\frac{p}{q})^2},$$

since the sum in $|S_1|$ is the same as Σ_2 in the proof of Theorem 1.1.

Next, we bound S_2 from above. In the same manner as for $|S_2(T,\phi)|$ in the proof of Theorem 1.1 we obtain that

$$S_2 \ll T(\log T)^{(\frac{p}{q})^2}.$$

Altogether it follows that

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^{2k} dt \gg T(\log T)^{k^2}$$

holds for any rational number $k \geq 1$.

6. Remark

As a consequence of Proposition 4.1 we have for any non-negative integer l and any non-negative real k

$$\max_{\phi \in [0,\pi)} \sum_{T \leq t_n(\phi) \leq 2T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it_n(\phi)\right)\right|^{2k} \gg \log T \int_T^{2T} \left|\zeta^{(l)}\left(\tfrac{1}{2}+it\right)\right|^{2k} dt.$$

Now, using the unconditional lower bound for $I_{k,0}(T)$ by Heath-Brown [7], resp. the conditional one by Ramachandra [16], we can deduce that

$$\max_{\phi \in [0,\pi)} \sum_{T \leq t_n(\phi) \leq 2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} \gg T(\log T)^{k^2+1}.$$

holds for any rational $k \geq 0$, resp. under the assumption of the Riemann hypothesis for any real $k \geq 0$.

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