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Equations for Mahler measure and isogenies

par Matilde N. LALÍN

Résumé. Nous étudions quelques équations fonctionnelles de la mesure de Mahler de familles de courbes de genre 1 en utilisant des isogénies entre les courbes. Ces équations ont le potentiel d'aider à trouver des relations entre la mesure de Mahler et des valeurs spéciales de fonctions L. Ces notes sont inspirées d'une présentation de l'auteure aux ≪Cuartas Jornadas de Teoría de Números≫, à Bilbao, 2011.

Abstract. We study some functional equations between Mahler measures of genus-one curves in terms of isogenies between the curves. These equations have the potential to establish relationships between Mahler measure and especial values of L-functions. These notes are based on a talk that the author gave at the "Cuartas Jornadas de Teoría de Números", Bilbao, 2011.

1. Introduction

The (logarithmic) Mahler measure of a nonzero polynomial $P(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ is defined by

$$\mathrm{m}\left(P(x_1,\dots,x_n)\right) = \int_0^1 \cdots \int_0^1 \log\left|P\left(\mathrm{e}^{2\pi i\theta_1},\dots,\mathrm{e}^{2\pi i\theta_n}\right)\right| d\theta_1 \dots d\theta_n.$$

Boyd [5] studied the family of two-variable polynomials $P_\alpha(x,y) = x + \frac{1}{x} + y + \frac{1}{y} + \alpha$ and found numerically that

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+\alpha\right) \stackrel{?}{=} \frac{L'(E_{(\alpha)},0)}{s_\alpha} \qquad \alpha\in\mathbb{N}, \alpha\neq 0,4 \tag{1.1}$$

where s_α is a rational number (often integer), and $E_{(\alpha)}$ is the elliptic curve which corresponds to the zero set of the polynomial. When $\alpha = 4$ the curve has genus zero and a formula with a Dirichlet L-function is easily proved.

Boyd studied numerically several other families of curves, including the following:

$$g(\alpha) := \mathrm{m}\left((1+x)(1+y)(x+y)-\alpha xy\right),$$
$$n(\alpha) := \mathrm{m}\left(x^3+y^3+1-\alpha xy\right),$$
$$r(\alpha) := \mathrm{m}\left((1+x)(1+y)(1+x+y)-\alpha xy\right).$$

At the same time as Boyd's numerical results and analysis, Deninger [9] made the connection between these formulas and Beilinson's conjectures. Parallel to this, Rodriguez-Villegas [15] made a more detailed analysis of Boyd's results in the light of Beilinson's conjectures.

Since then, results in the direction of equations (1.1) have been proven for $m(\alpha), g(\alpha), n(\alpha)$, and $r(\alpha)$ by Rodriguez–Villegas [15], Brunault [6, 7], Mellit [14], and Rogers and Zudilin [18, 19]. For example, Rogers and Zudilin proved in [18] that

$$m(8) = \frac{24}{\pi^2} L(E_{24}, 2),$$
$$g(4) = \frac{3}{4} n\left(\sqrt[3]{32}\right) = \frac{10}{\pi^2} L(E_{20}, 2),$$

where E_k denotes an elliptic curve of conductor k.

In all cases, the strategy consists on reducing or proving particular cases of Beilinson's conjectures.

Many efforts have been also devoted to establishing identities between Mahler measures of different polynomials in the previous families. These efforts originated as intermediate steps towards the proofs of the previously discussed identities, and they led to results that are interesting in their own right.

The first identity of such kind was proved by Rodriguez-Villegas [16]

$$\mathrm{m}(y^2 + 2xy + y - x^3 - 2x^2 - x) = \frac{5}{7}\mathrm{m}(y^2 + 4xy + y - x^3 + x^2),$$

by translating this to the language of regulators. Following the same techniques, similar results were given by Bertin [2, 3], Touafek and Kerada [23], Lalín and Rogers [13], Lalín [12], and Guillera and Rogers [10]. Some of these proofs rely on functional equations involving $m(\alpha)$ or the other functions.

Our goal here is to prove some old and new functional identities by relating them to isogenies between the corresponding curves. Our results are as follows.

Theorem 1.1. *For $p > 0$, we have*

$$g(4p(1+p)) + g\left(\frac{4(1+p)}{p^2}\right) = 2g\left(\frac{2(1+p)^2}{p}\right), \tag{1.2}$$

and

$$g\left(-\frac{2p^2}{(1+p)}\right) + g\left(\frac{4(1+p)}{p^2}\right) = g\left(\frac{2(1+p)^2}{p}\right) + g\left(-\frac{2}{p(1+p)}\right). \tag{1.3}$$

For $|p| > 8$, we have,

$$g(p) = \frac{1}{3}n\left(\frac{p+4}{p^{2/3}}\right) + \frac{4}{3}n\left(\frac{p-2}{p^{1/3}}\right). \tag{1.4}$$

Equation (1.2) was originally proved in Theorem 6 of [18] for $\frac{\sqrt{3}-1}{2} \leq p \leq 1$ while equation (1.4) is formula (2.27) in Theorem 2.4 of [13], where it was proved for $|p|$ sufficiently large. Equation (1.3) seems to be completely new.

$p = -8$ yields a limiting case in equation (1.4) that leads to

$$n(-1) + 4n(5) = 3g(-8),$$

an identity known to Mellit [14].

We are going to prove such results by finding relationships between regulators via pull backs through 2− and 3−isogenies.

2. The relationship with the regulator

Let $E/\mathbb{C}$ be an elliptic curve. The regulator map can be defined by

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$
$$\{x, y\} \to \left\{\gamma \to \int_\gamma \eta(x, y)\right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$ and

$$\eta(x, y) := \log |x| \mathrm{d} \arg y - \log |y| \mathrm{d} \arg x$$

Here we think of $H^1(E, \mathbb{R})$ as the dual of $H_1(E, \mathbb{Z})$. The function is well defined because $\eta(x, 1 - x) = \mathrm{d}D(x)$ (and therefore it is exact). Here

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1 - z) \log |z|$$

is the Bloch-Wigner dilogarithm.

The last ingredient in this picture is $K_2(E)$. Under certain conditions that are verified in our cases, we can think of $K_2(E) \otimes \mathbb{Q} \subset K_2(\mathbb{C}(E)) \otimes \mathbb{Q}$ and

$$K_2(\mathbb{C}(E)) \cong \Lambda^2 \mathbb{C}(E)^\times / \{x \otimes (1 - x)\}$$

by Matsumoto's Theorem.

Assume that E is defined over $\mathbb{R}$. Because of the way that complex conjugation acts on η, the regulator map is trivial for the classes in $H_1(E, \mathbb{Z})^+$, the cycles that remain invariant by complex conjugation. Therefore it suffices to consider the regulator as a function on $H_1(E, \mathbb{Z})^-$.

After the works of Deninger [9] and Rodriguez-Villegas [15], we write

$$\mathrm{m}(P) = \frac{1}{2\pi} r(\{x, y\})[\{(x, y), y = P(x), |x| = |y| = 1\}]. \tag{2.1}$$

This is easily accomplished by Jensen's formula.

We write $E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ and let $\beta = a + b\tau \in E(\mathbb{C})$. Then Bloch [4] defines the regulator function by a Kronecker-Eisenstein series

$$R_\tau(\beta) = -\frac{y_\tau^2}{\pi} \sum_{m,n \in \mathbb{Z}}{}' \frac{\sin(2\pi(an - bm))}{(m\tau + n)^2 (m\bar{\tau} + n)}, \tag{2.2}$$

where y_τ is the imaginary part of τ. Then

$$2\int_E \eta(x,y)\wedge\omega = R_\tau((x)\diamond(y)).$$

Here the diamond operation $\diamond : \mathbb{C}(E)^*\otimes\mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$ is defined on the divisors (x) and (y) as

$$(x)\diamond(y) = \sum m_i n_j (a_i - b_j).$$

for

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

(See [15] for more details.)

Mellit follows the details of equation (2.1) for the families we want to consider and proves, more specifically, that

$$g(\alpha) = \frac{c_\alpha}{2\pi} R((x)\diamond(y)) \qquad c_\alpha = \begin{cases} -1 & \text{if } \alpha > 0, \alpha \neq 8 \\ 1 & \text{if } \alpha < 0, \alpha \neq -1 \end{cases}, \tag{2.3}$$

$$n(\alpha) = -\frac{d_\alpha}{2\pi} R((x)\diamond(y)) \qquad d_\alpha = \begin{cases} 1 & \text{if } \alpha > 3 \\ -2 & \text{if } \alpha \leq -1 \end{cases}. \tag{2.4}$$

These results are listed in section 2.2 from [14]. Our coefficients are slightly different because we use a different normalization for the $\diamond$ operation.

3. Functional identities involving $g(\alpha)$ and $n(\alpha)$

3.1. The case of $g(\alpha)$. Recall that the family that defines $g(\alpha)$ corresponds to $\Gamma_0(6)$. The following form of the equation will be more convenient for our computations:

$$(x+1)y^2 + (x^2 + (2-\alpha)x + 1)y + (x^2 + x).$$

If we let $k = 2-\alpha$, then we obtain the Weierstrass form

$$Y^2 = X(X^2 + (k^2-12)X - 16(k-3)). \tag{3.1}$$

The change of variables is given by

$$X = -\frac{4x(x+y+1)}{y},$$

$$Y = -\frac{4x(2xy+4y-ky+2x^2+xk+k-2)}{y},$$

$$x = \frac{Y-(k-2)X}{2(X+4(k-3))},$$

$$y = \frac{-XY - 4(k-1)(k-3)Y + (3k-8)X^2 + 4(k-3)^2(k+2)X - 32(k-3)^2}{(X+4(k-3))(Y-(k-4)X - 8(3-k))}.$$

This family has always a point of order 6 which we will denote by

$$P = (4(3-k), 4(3-k)(2-k))$$

in the (X, Y) coordinates. Then $2P = (4, 4(2-k))$, $3P = (0,0)$, $4P = (4, 4(k-2))$, and $5P = (4(3-k), -4(3-k)(2-k))$. Sometimes, when working with more than one curve at the same time, we will write P_α to specify that we are discussing the P from the curve with parameter α.

The divisors of x and y are supported in the torsion group.

$$\begin{aligned}(x) &= ((3P)+(4P)+(5P)-3O)-((P)+(5P)-2O)\\ &= (3P)+(4P)-(P)-O,\\ (y) &= ((4P)+4(5P)-5O)-((P)+(5P)-2O)-((2P)+2(5P)-3O)\\ &= (4P)+(5P)-(P)-(2P).\end{aligned}$$

Applying the diamond operation, we have

$$(x)\diamond(y) = -6(P)-6(2P).$$

We are going to consider identities involving the parameter $\alpha_0 = \frac{2(1+p)^2}{p}$. This particular curve $E_{\left(\frac{2(1+p)^2}{p}\right)}$ has, in addition to the group generated by P, two more points of order 2, namely $Q = (-4p(p+2), 0)$, and $3P+Q = \left(-\frac{4(1+2p)}{p^2}, 0\right)$. Then $P+Q = \left(-4(2p+1), \frac{8(2p+1)(1-p^2)}{p}\right)$ and $2P+Q = \left(-\frac{4(p+2)}{p}, \frac{8(p+2)(1-p^2)}{p^2}\right)$.

Now set $X = \bar{X}2(k-2)+4$ and $Y = \bar{Y}4(k-2)$. We have,

$$\bar{Y}^2 = \left(\bar{X}\frac{k-2}{2}+1\right)\left(\bar{X}^2+\frac{k+2}{2}\bar{X}+1\right).$$

We consider some specific values for k.

For $k = 2 - \frac{2(1+p)^2}{p} = -\frac{2(p^2+p+1)}{p}$.

$$\begin{aligned}\bar{Y}^2 &= \left(-\frac{(1+p)^2}{p}\bar{X}+1\right)\left(\bar{X}^2-\frac{p^2+1}{p}\bar{X}+1\right)\\ &= \left(-\frac{(1+p)^2}{p}\bar{X}+1\right)(\bar{X}-p)\left(\bar{X}-\frac{1}{p}\right).\end{aligned}$$

For $k = 2-4p(1+p)$

$$\bar{Y}^2 = \left(-2p(1+p)\bar{X}+1\right)\left(\bar{X}^2-2(p^2+p-1)\bar{X}+1\right).$$

For $k = 2 - \frac{4(1+p)}{p^2}$

$$\bar{Y}^2 = \left(-\frac{2(1+p)}{p^2}\bar{X}+1\right)\left(\bar{X}^2+\frac{2(p^2-p-1)}{p^2}\bar{X}+1\right).$$

We can explicitly write the isogenies at the level of the Weierstrass models. By using the well-known isogeny of degree 2 (see for example [8], chapter 14, or [20], page 74, example 4.5):

$$\phi : \{E : Y^2 = X(X^2 + aX + b)\} \to \{\widehat{E} : \hat{y}^2 = \hat{X}(\hat{X}^2 - 2a\hat{X} + (a^2 - 4b))\}$$

given by

$$(X, Y) \to \left(\frac{Y^2}{X^2}, \frac{Y(b - X^2)}{X^2} \right)$$

(as always, we require that $a^2 - 4b \neq 0$).

We have, in terms of $(\bar{X}, \bar{Y})$,

$$\phi_1 : E_{\frac{2(1+p)^2}{p}} \to E_{4p(1+p)}, \qquad \phi_2 : E_{\frac{2(1+p)^2}{p}} \to E_{\frac{4(1+p)}{p^2}},$$

$$\phi_1 : \left(\bar{X}, \bar{Y}\right) \to \left(\frac{\bar{X}((1+p)\bar{X} - 2)}{2\left(\bar{X} - p\right)}, \frac{\bar{Y}p\left((1+p)\bar{X}^2 - 2p(1+p)\bar{X} + 2p\right)}{2\left(\bar{X} - p\right)^2} \right),$$

$$\phi_2 : \left(\bar{X}, \bar{Y}\right) \to \left(\frac{\bar{X}((1+p)\bar{X} - 2p)}{2\left(p\bar{X} - 1\right)}, \frac{\bar{Y}\left((1+p)\bar{X}^2 - \frac{2(1+p)}{p}\bar{X} + 2\right)}{2\left(p\bar{X} - 1\right)^2} \right).$$

Let us write $\alpha_1 = 4p(1+p)$ and $\alpha_2 = \frac{4(1+p)}{p^2}$. We have

$$\phi_1(P_{\alpha_0}) = P_{\alpha_1}, \quad \ker(\phi_1) = \{Q_{\alpha_0}, O_{\alpha_0}\},$$
$$\phi_2(P_{\alpha_0}) = P_{\alpha_2}, \quad \ker(\phi_2) = \{Q_{\alpha_0} + 3P_{\alpha_0}, O_{\alpha_0}\}.$$

Using this, we find that

$$(x_{\alpha_1} \circ \phi_1) \diamond (y_{\alpha_1} \circ \phi_1) = -12((P_{\alpha_0}) + (2P_{\alpha_0}) + (P_{\alpha_0} + Q_{\alpha_0}) + (2P_{\alpha_0} + Q_{\alpha_0})),$$

and

$$(x_{\alpha_2} \circ \phi_2) \diamond (y_{\alpha_2} \circ \phi_2) = -12((P_{\alpha_0}) + (2P_{\alpha_0}) - (P_{\alpha_0} + Q_{\alpha_0}) - (2P_{\alpha_0} + Q_{\alpha_0})).$$

These computations imply that

$$\frac{1}{2}R_{(\alpha_0)}\left((x_{\alpha_1} \circ \phi_1) \diamond (y_{\alpha_1} \circ \phi_1)\right) + \frac{1}{2}R_{(\alpha_0)}\left((x_{\alpha_2} \circ \phi_2) \diamond (y_{\alpha_2} \circ \phi_2)\right)$$
$$= 2R_{(\alpha_0)}\left((x_{\alpha_0}) \diamond (y_{\alpha_0})\right).$$

It is proven in [14] that the integration path for the regulator is $[0, \tau]$ for $\alpha > 0$. It is easy to see that $\alpha_0 > 8$ for $p > 0, p \neq 1$. Also, $\tau \in i\mathbb{R}$ if $8 < \alpha$ and $\tau \in \frac{1}{2} + i\mathbb{R}$ if $0 < \alpha_0 < 8$. If the periods are $\{\omega_1, \omega_2\}$ with $\omega_1 \in \mathbb{R}$, we obtain that P_α corresponds to $\frac{5}{6}\omega_1$ (for $0 < \alpha$) and Q_α to $\frac{1}{2}\omega_2$ (for $8 < \alpha$). Now suppose that $p > 1$. Then $\alpha_1 > 8$ and $\alpha_2 < 8$. We notice that the path $[0, \tau_{\alpha_0}]$ is sent to $[0, 2\tau_{\alpha_1}]$ by ϕ_1 (respectively $\left[0, 2\left(\tau_{\alpha_2} - \frac{1}{2}\right)\right]$ by ϕ_2). Thus $R_{(\alpha_1)}\left((x_{\alpha_1}) \diamond (y_{\alpha_1})\right) = \frac{1}{2}R_{(\alpha_0)}\left((x_{\alpha_1} \circ \phi_1) \diamond (y_{\alpha_1} \circ \phi_1)\right)$ and similarly for α_2.

Therefore

$$R_{(\alpha_1)}\left((x_{\alpha_1})\diamond(y_{\alpha_1})\right)+R_{(\alpha_2)}\left((x_{\alpha_2})\diamond(y_{\alpha_2})\right)=2R_{(\alpha_0)}\left((x_{\alpha_0})\diamond(y_{\alpha_0})\right),$$

which proves equation (1.2):

$$g(4p(1+p))+g\left(\frac{4(1+p)}{p^2}\right)=2g\left(\frac{2(1+p)^2}{p}\right).$$

Notice that exchanging p by $\frac{1}{p}$ will only exchange the order of the terms in the left, and that the equation is trivially true for $p=1$. Therefore, it is valid for $p>0$.

We now consider the following isomorphisms,

$$\varphi_1: E_{\left(\frac{2(1+p)^2}{p}\right)}\to E_{\left(-\frac{2}{p(1+p)}\right)},\qquad \varphi_2: E_{\left(\frac{2(1+p)^2}{p}\right)}\to E_{\left(-\frac{2p^2}{(1+p)}\right)},$$

given by

$$\begin{aligned}\varphi_1:(X,Y)&\to\left(\frac{X+4p(2+p)}{(p+1)^2},\frac{Y}{(p+1)^3}\right),\\ \varphi_2:(X,Y)&\to\left(\frac{p^2X+4(2p+1)}{(p+1)^2},\frac{p^3Y}{(p+1)^3}\right).\end{aligned}$$

Let $\beta_1=-\frac{2}{p(1+p)}$, and $\beta_2=-\frac{2p^2}{(1+p)}$. Then

$$\begin{aligned}\varphi_1(2P_{\alpha_0}+Q_{\alpha_0})&=P_{\beta_1},\\ \varphi_2(Q_{\alpha_0}-P_{\alpha_0})&=P_{\beta_2}.\end{aligned}$$

This means that

$$\begin{aligned}(x_{\beta_1}\circ\varphi_1)\diamond(y_{\beta_1}\circ\varphi_1)&=-6(2P_{\alpha_0}+Q_{\alpha_0})-6(-2P_{\alpha_0})\\ &=-6((2P_{\alpha_0}+Q_{\alpha_0})-(2P_{\alpha_0})),\\ (x_{\beta_2}\circ\varphi_2)\diamond(y_{\beta_2}\circ\varphi_2)&=-6(Q_{\alpha_0}-P_{\alpha_0})-6(-2P_{\alpha_0})\\ &=6((P_{\alpha_0}+Q_{\alpha_0})+(2P_{\alpha_0})).\end{aligned}$$

This translates into

$$\begin{aligned}&R_{(\alpha_0)}((x_{\beta_1}\circ\varphi_1)\diamond(y_{\beta_1}\circ\varphi_1))+\frac{1}{2}R_{(\alpha_0)}((x_{\alpha_2}\circ\phi_2)\diamond(y_{\alpha_2}\circ\phi_2))\\ &\qquad=R_{(\alpha_0)}((x_{\alpha_0})\diamond(y_{\alpha_0}))+R_{(\alpha_0)}((x_{\beta_2}\circ\varphi_2)\diamond(y_{\beta_2}\circ\varphi_2)).\end{aligned}$$

Using that $R_{(\alpha_2)}\left((x_{\alpha_2})\diamond(y_{\alpha_2})\right)=\frac{1}{2}R_{(\alpha_0)}\left((x_{\alpha_2}\circ\phi_2)\diamond(y_{\alpha_2}\circ\phi_2)\right)$

$$\begin{aligned}&R_{(\beta_1)}((x_{\beta_1})\diamond(y_{\beta_1}))+R_{(\alpha_2)}((x_{\alpha_2})\diamond(y_{\alpha_2}))\\ &\qquad=R_{(\alpha_0)}((x_{\alpha_0})\diamond(y_{\alpha_0}))+R_{(\beta_2)}((x_{\beta_2})\diamond(y_{\beta_2})).\end{aligned}$$

At the level of Mahler measure, this equation becomes, for $p>0$,

$$-g\left(-\frac{2}{p(1+p)}\right)+g\left(\frac{4(1+p)}{p^2}\right)=g\left(\frac{2(1+p)^2}{p}\right)-g\left(-\frac{2p^2}{(1+p)}\right),$$

which concludes the proof of equation (1.3).

In addition, notice that combining both (1.2), and (1.3), we also conclude

$$-g\left(-\frac{2}{p(1+p)}\right)+g\left(\frac{2(1+p)^2}{p}\right)=g\left(4p(1+p)\right)-g\left(-\frac{2p^2}{(1+p)}\right).$$

3.2. The case of $n(\alpha)$. Recall that $n(\alpha)$ corresponds to the Hesse family:

$$x^3+y^3+1-\alpha xy.$$

If we let $\alpha=\frac{3}{a}$, a Weierstrass form is given by

$$Y^2=X^3-27X^2+216(1-a^3)X-432(1-a^3)^2,$$

where the change of variables is given by

$$X=-\frac{12\left(a^3-1\right)\frac{x}{a}}{\frac{x}{a}+y+1},\qquad Y=\frac{36\left(a^3-1\right)(y-1)}{\frac{x}{a}+y+1},$$

$$x=-\frac{6aX}{3X+36(a^3-1)-Y},\qquad y=\frac{3X+36(a^3-1)+Y}{3X+36(a^3-1)-Y}.$$

We have the following points of order 3 over $\bar{\mathbb{Q}}(a)$:

$$\begin{aligned}
Q&=(0,12(1-\alpha)\sqrt{3}i),\\
A&=(6(a-1)(-(a+2)+a\sqrt{3}i),18a(a-1)(2a+1+\sqrt{3}i)),\\
B&=(6(a-1)(-(a+2)-a\sqrt{3}i),18a(a-1)(2a+1-\sqrt{3}i)).
\end{aligned}$$

These points satisfy $Q+A=B$.

We proceed to compute the divisors of x and y:

$$\begin{aligned}
(x)&=((Q)+(-Q)-2O)-((-A-B)+(A)+(B)-3O)\\
&=(Q)+(-Q)+O-(-A-B)-(A)-(B),\\
(y)&=((A+B)+(-A)+(-B))-((-A-B)+(A)+(B)).
\end{aligned}$$

The diamond operation yields

$$\begin{aligned}
(x)\diamond(y)=&\,2(Q-A-B)+2(Q+A)+2(Q+B)-2(Q+A+B)\\
&-2(Q-A)-2(Q-B)-(A+B)-(2A)-(2B)-2(-B)\\
&-2(-A)-2(A+B)-2(A+B)+2(A)+2(B)\\
=&-9(A+B)+9(A)+9(B).
\end{aligned}$$

Notice that the Weirstrass equation can be written as

$$F_{(\alpha)} : Y^2 = X^3 - 27(X - 4(1 - a^3))^2.$$

Thus, for $\alpha = \frac{p-2}{p^{1/3}}$, we obtain

$$F_{\left(\frac{p-2}{p^{1/3}}\right)} : Y^2 = X^3 - 27\left(X - 4\left(1 - \frac{27p}{(p-2)^3}\right)\right)^2.$$

For $\alpha = \frac{p+4}{p^{2/3}}$, we get

$$F_{\left(\frac{p+4}{p^{2/3}}\right)} : Y^2 = X^3 - 27\left(X - 4\left(1 - \frac{27p^2}{(p+4)^3}\right)\right)^2.$$

We now go back to the $g(\alpha)$ family from equation (3.1) with $\alpha = p$:

$$Y^2 = X^3 + (p^2 - 4p - 8)X^2 + 16(p+1)X.$$

If we let $X = \bar{X}(p-2)^2 + 4$ and $Y = \bar{Y}(p-2)^3$,

$$E_{(p)} : \bar{Y}^2 = \bar{X}^3 + \left(\bar{X} + \frac{4p}{(p-2)^3}\right)^2.$$

We now use the degree-3 isogeny (see Top [22])

$$\psi : \{E : Y^2 = X^3 + a(X-b)^2\} \to \{\widehat{E} : \hat{Y}^2 = \hat{X}^3 - 27a(\hat{X} - 4a - 27b)^2\}$$

given by

$$(X, Y) \to \left(\frac{3(6Y^2 + 6ab^2 - 3X^3 - 2aX^2)}{X^2}, \frac{27Y(-4abX + 8ab^2 - X^3)}{X^3}\right).$$

We find

$$\psi_1 : E_{(p)} \to F_{\left(\frac{p-2}{p^{1/3}}\right)}$$

$$\psi_1 : \left(\bar{X}, \bar{Y}\right) \to \left(\frac{3\left(6\bar{Y}^2 + \frac{96p^2}{(p-2)^6} - 3\bar{X}^3 - 2\bar{X}^2\right)}{\bar{X}^2}, \frac{27\bar{Y}\left(\frac{16p}{(p-2)^3}\bar{X} + \frac{128p^2}{(p-2)^6} - \bar{X}^3\right)}{\bar{X}^3}\right).$$

In order to obtain $F_{\left(\frac{p+4}{p^{2/3}}\right)}$ we need a degree 6 isogeny. Thus we apply first the usual 2-isogeny

$$\phi_3 : E_{(p)} \to E_{\left(\frac{-8}{p}\right)} : \bar{Y}^2 = \bar{X}^3 + \left(\bar{X} + \frac{4p^2}{(p+4)^3}\right)^2,$$

$$\phi_3 : (\bar{X}, \bar{Y}) \to \left(\frac{(p-2)^2 \bar{X}(\bar{X}(p-2)^2 - 4p)}{(\bar{X}(p-2)^2+4)(p+4)^2}, \right.$$
$$\left. - \frac{(p-2)^3 \bar{Y}(16(p+1) - (\bar{X}(p-2)^2+4)^2)}{(\bar{X}(p-2)^2+4)^2(p+4)^3} \right),$$

and then the 3-isogeny that we have just described

$$\psi_2 : E_{\left(\frac{-8}{p}\right)} \to F_{\left(\frac{p+4}{p^{2/3}}\right)}.$$

$$\psi_2 : (\bar{X}, \bar{Y}) \to$$
$$\left(\frac{3(6\bar{Y}^2 + \frac{96p^4}{(p+4)^6} - 3\bar{X}^3 - 2\bar{X}^2)}{\bar{X}^2}, \frac{27\bar{Y}(\frac{16p^2}{(p+4)^3}\bar{X} + \frac{128p^4}{(p+4)^6} - \bar{X}^3)}{\bar{X}^3} \right).$$

Combining both isogenies, we obtain

$$\psi_2 \circ \phi_3 : E_{(p)} \to F_{\left(\frac{p+4}{p^{2/3}}\right)}.$$

Let us write $\gamma_1 = \frac{p-2}{p^{1/3}}$ and $\gamma_2 = \frac{p+4}{p^{2/3}}$. It follows that

$$\psi_1(\tilde{A}) = A_{\gamma_1}, \psi_1(\tilde{B}) = B_{\gamma_1} \quad \ker(\psi_1) = \langle 2P_p \rangle$$

for some $\tilde{A}$, $\tilde{B}$ of order dividing 9 in $E_{(p)}$. Since the image of $3\tilde{A}$ is $3A_{\gamma_1} = O_{\gamma_1}$, we conclude that $3\tilde{A} = O_p, 2P_p$ or $4P_p$ (and similarly for $\tilde{B}$). From the formula, it is easy to see that the preimage of Q_{γ_1} (given by $\tilde{B} - \tilde{A}$, $\tilde{B} - \tilde{A} + 2P_p$, $\tilde{B} - \tilde{A} + 4P_p$) are points of order 3 (see, for example, [21], page 40, point (c)). Then the 3-torsion group (over the algebraic closure) is generated by $2P_p$ and $\tilde{B} - \tilde{A}$. From this, we conclude that $3\tilde{A} = 3\tilde{B} = 2P_p$ or $4P_p$. Using this information, and the divisors (x_{γ_1}) and (y_{γ_1}) in $F_{\left(\frac{p-2}{p^{1/3}}\right)}$, we can compute $(x_{\gamma_1} \circ \psi_1) \diamond (y_{\gamma_1} \circ \psi_1)$. We find that

$$(x_{\gamma_1} \circ \psi_1) \diamond (y_{\gamma_1} \circ \psi_1) = 27((\tilde{A}) + (\tilde{A} + 2P_p) + (\tilde{A} + 4P_p)$$
$$+ (\tilde{B}) + (\tilde{B} + 2P_p) + (\tilde{B} + 4P_p) - (\tilde{A} + \tilde{B})$$
$$- (\tilde{A} + \tilde{B} + 2P_p) - (\tilde{A} + \tilde{B} + 4P_p)).$$

Since the relation between $F_{\left(\frac{p-2}{p^{1/3}}\right)}$ and $F_{\left(\frac{p+4}{p^{2/3}}\right)}$ is given by $p \to -\frac{8}{p}$, it is also clear that $\psi_2 \circ \phi_3(\tilde{A}) = A_{\gamma_2}$ and analogously for the B's. This is summarized as follows:

$$\psi_2 \circ \phi_3(\tilde{A}) = A_{\gamma_2}, \psi_2 \circ \phi_3(\tilde{B}) = B_{\gamma_2}, \quad \ker(\psi_2 \circ \phi_3) = \langle P_p \rangle .$$

Thus

$$
\begin{aligned}
&(x_{\gamma_2}\circ\psi_2\circ\phi_3)\diamond(y_{\gamma_2}\circ\psi_2\circ\phi_3)=\\
&\quad 54((\tilde{A})+(\tilde{A}+2P_p)+(\tilde{A}+4P_p)+(\tilde{B})+(\tilde{B}+2P_p)\\
&\qquad +(\tilde{B}+4P_p)-(\tilde{A}+\tilde{B})-(\tilde{A}+\tilde{B}+2P_p)-(\tilde{A}+\tilde{B}+4P_p)\\
&\qquad +(\tilde{A}+P_p)+(\tilde{A}+3P_p)+(\tilde{A}+5P_p)+(\tilde{B}+P_p)+(\tilde{B}+3P_p)\\
&\qquad +(\tilde{B}+5P_p)-(\tilde{A}+\tilde{B}+P_p)-(\tilde{A}+\tilde{B}+3P_p)-(\tilde{A}+\tilde{B}+5P_p)).
\end{aligned}
$$

Now consider $\widehat{\psi_1}\circ\psi_1 : E_{(p)}\to E_{(p)}$ corresponding to multiplication by 3. We have that

$$[3](\pm(P_p-\tilde{A}))=P_p,\quad [3](\pm\tilde{A})=2P_p,\quad \ker([3])=\left\langle 2P_p,\tilde{A}-\tilde{B}\right\rangle.$$

Here the sign $\pm$ corresponds to the sign of $3\tilde{A}=\pm 2P_p$.

Thus

$$
\begin{aligned}
&(x_p\circ[3])\diamond(y_p\circ[3])=\\
&\quad \mp 54((\tilde{A})+(\tilde{A}+2P_p)+(\tilde{A}+4P_p)+(\tilde{B})+(\tilde{B}+2P_p)\\
&\qquad +(\tilde{B}+4P_p)-(\tilde{A}+\tilde{B})-(\tilde{A}+\tilde{B}+2P_p)-(\tilde{A}+\tilde{B}+4P_p)\\
&\qquad +(\tilde{A}+P_p)+(\tilde{A}+3P_p)+(\tilde{A}+5P_p)+(\tilde{B}+P_p)+(\tilde{B}+3P_p)\\
&\qquad +(\tilde{B}+5P_p)-(\tilde{A}+\tilde{B}+P_p)-(\tilde{A}+\tilde{B}+3P_p)-(\tilde{A}+\tilde{B}+5P_p)).
\end{aligned}
$$

These computations show that

$$
\begin{aligned}
4R_{(p)}\left((x_{\gamma_1}\circ\psi_1)\diamond(y_{\gamma_1}\circ\psi_1)\right)-R_{(p)}\left((x_{\gamma_2}\circ\psi_2\circ\phi_3)\diamond(y_{\gamma_2}\circ\psi_2\circ\phi_3)\right)&\\
=\mp R_{(p)}\left((x_p\circ[3])\diamond(y_p\circ[3])\right).&
\end{aligned}
$$

The integration path for the regulator in the family F is proven in [14] to be $[0,\tau]$ while $\tau\in i\mathbb{R}$ for $\alpha>3$ and $[0,2\tau-1]$ while $\tau\in\frac{1}{2}+i\mathbb{R}$ for $\alpha<-1$. Notice that $p>8$ implies that $\gamma_1,\gamma_2>3$. Now ϕ_3 sends $[0,\tau_p]$ to $\left[0,\tau_{\frac{-8}{p}}\right]$ (as it annihilates $3P$ which is related to the real period) but changes its orientation. Therefore, $R_{\left(\frac{-8}{p}\right)}((x_{\frac{-8}{p}})\diamond(y_{\frac{-8}{p}}))=-R_{(p)}((x_{\frac{-8}{p}}\circ\psi_2)\diamond(y_{\frac{-8}{p}}\circ\psi_2))$. By the same idea, $R_{(\gamma_1)}((x_{\gamma_1})\diamond(y_{\gamma_1}))=R_{(p)}((x_{\gamma_1}\circ\psi_1)\diamond(y_{\gamma_1}\circ\psi_1))$, since ψ_1 annihilates $2P$, and thus $R_{(\gamma_2)}((x_{\gamma_2})\diamond(y_{\gamma_2}))=-R_{(p)}((x_{\gamma_2}\circ\psi_2\circ\phi_3)\diamond(y_{\gamma_2}\circ\psi_2\circ\phi_3))$. Similarly it is clear that $R_{(p)}((x_p)\diamond(y_p))=\frac{1}{3}R_{(p)}((x_p\circ[3])\diamond(y_p\circ[3]))$.

Therefore, we obtain,

$$4R_{(\gamma_1)}((x_{\gamma_1})\diamond(y_{\gamma_1}))+R_{(\gamma_2)}((x_{\gamma_2})\diamond(y_{\gamma_2}))=\mp 3R_{(p)}((x_p\diamond(y_p)). \tag{3.2}$$

If we take $p<-8$, we have that $\gamma_1>3$, $\gamma_2<-1$, and $0<\frac{-8}{p}<1$. A similar analysis on integration paths and orientations yields

$R_{(\gamma_2)}((x_{\gamma_2}) \diamond (y_{\gamma_2})) = \frac{1}{2} R_{(p)}((x_{\gamma_2} \circ \psi_2 \circ \phi_3) \diamond (y_{\gamma_2} \circ \psi_2 \circ \phi_3))$, and

$$4R_{(\gamma_1)}((x_{\gamma_1}) \diamond (y_{\gamma_1})) - 2R_{(\gamma_2)}((x_{\gamma_2}) \diamond (y_{\gamma_2})) = \mp 3R_{(p)}((x_p \diamond (y_p)). \tag{3.3}$$

Combining equations (3.2) and (3.3) with equations (2.3) and (2.4), and taking into account that the terms must all be positive, we get equation (1.4):

$$\frac{4}{3} n \left(\frac{p-2}{p^{1/3}} \right) + \frac{1}{3} n \left(\frac{p+4}{p^{2/3}} \right) = g(p).$$

This concludes the proof of the result.

4. Conclusion

We have proved some functional identities involving Mahler measures of families of genus-one curves by using relations between regulators and isogenies. This is not the only method to prove such formulas, which can be also attacked by means of hypergeometric functions. It would be interesting to explore other proofs and see if the conditions on the parameter p can be relaxed.

It would be also interesting to see if these formulas can be used to prove new formulas between Mahler measures and special values of L-functions of elliptic curves at $s = 2$.

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