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Ergodic Universality Theorems for the Riemann Zeta-Function and other *L*-Functions

par Jörn STEUDING

Résumé. Nous prouvons un nouveau type de théorème d'universalité pour la fonction zêta de Riemann et d'autres fonctions L (qui sont universelles au sens du théorème de Voronin). Contrairement aux théorèmes d'universalité précédents pour la fonction zêta ou ses généralisations diverses, ici les approximations sont obtenues à partir de l'orbite d'une transformation ergodique sur la droite réelle.

Abstract. We prove a new type of universality theorem for the Riemann zeta-function and other L-functions (which are universal in the sense of Voronin's theorem). In contrast to previous universality theorems for the zeta-function or its various generalizations, here the approximating shifts are taken from the orbit of an ergodic transformation on the real line.

1. Motivation and Statement of The Main Theorem

In 1975, Voronin [7] proved that the Riemann zeta-function $\zeta(s)$ possesses the following remarkable approximation property: *Let* $0 < r < \frac{1}{4}$ *and suppose that* $f(s)$ *is a continuous function on the disk* $|s| \leq r$ *which is analytic and non-vanishing in the interior of the disk. Then, for any* $\epsilon > 0$, *there exists a real number* τ *such that*

$$\max_{|s|\leq r} |\zeta(s + \tfrac{3}{4} + i\tau) - f(s)| < \epsilon;$$

moreover, his proof showed that *the set of such* τ *has positive lower density* with respect to Lebesgue measure. This so-called universality theorem has been extended and generalized in various ways. For instance, Voronin [8] obtained a simultaneous approximation theorem for Dirichlet L-functions associated with non-equivalent characters; here two characters are said to be non-equivalent if they are not induced by the same character. The precise formulation of this so-called joint universality theorem is as follows: *Let* $\chi_1, \ldots, \chi_m$ *be pairwise non-equivalent Dirichlet characters,* $\mathcal{K}_1, \ldots, \mathcal{K}_m$ *be compact subsets of the strip* $\frac{1}{2} < \operatorname{Re} s < 1$ *with connected complements. Further, for each* $j = 1, \ldots, m$, *let* $f_j(s)$ *be a continuous function on* $\mathcal{K}_j$

which is analytic and non-vanishing in the interior of $\mathcal{K}_j$*. Then, for any* $\epsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{1\le j\le m} \max_{s\in\mathcal{K}_j} |L(s+i\tau,\chi_j) - f_j(s)| < \epsilon \right\} > 0.$$

Further examples of universal zeta- and L-functions are Dedekind zeta-functions $\zeta_{\mathbb{K}}(s)$ to a number field $\mathbb{K}$, L-functions associated with modular forms, and certain Hurwitz zeta-functions, to mention just a few examples. Reich [5] succeeded in proving so-called discrete universality theorems where the approximating shifts are taken from an arithmetic progression, e.g.,

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ n \in \mathbb{N} \cap [1,N] \,:\, \max_{s\in\mathcal{K}} |\zeta_{\mathbb{K}}(s+i\Delta n) - f(s)| < \epsilon \right\} > 0$$

for suitable target functions f defined on suitable sets $\mathcal{K}$; here Δ is an arbitrary fixed real number different from zero and $\sharp\mathcal{M}$ denotes the cardinality of the discrete set $\mathcal{M}$. For more information we refer to [5, 6].

Given a domain $\mathcal{D}$ we call a family of analytic functions $\mathcal{L}_1, \ldots, \mathcal{L}_m$ a jointly universal family with respect to $\mathcal{D}$ if, for any collection of compact subsets $\mathcal{K}_1, \ldots, \mathcal{K}_m$ of $\mathcal{D}$ with connected complements, any family of continuous functions f_j defined on $\mathcal{K}_j$, each of which is analytic and non-vanishing in the interior of $\mathcal{K}_j$, and any $\epsilon > 0$, there exists some real number $\tau > 0$ such that

$$\max_{1\le j\le m} \max_{s\in\mathcal{K}_j} |\mathcal{L}_j(s+i\tau) - f_j(s)| < \epsilon. \tag{1.1}$$

We shall say that such a family is strongly jointly universal with respect to $\mathcal{D}$ if the assumption on the non-vanishing of the target functions can be dropped (as, for example, for certain Hurwitz zeta-functions; see [3]).

In this note we shall investigate the phenomenon of universality on orbits of certain ergodic transformations. In order to state the main result we first recall some basic notions from ergodic theory.

Given a probability space $(X, \mathcal{F}, \mathsf{P})$ consisting of a non-empty space X, an associated σ-algebra $\mathcal{F}$, and a probability measure P on $\mathcal{F}$, a measurable transformation $\mathsf{T} : X \to X$ is said to be measure preserving with respect to P if $\mathsf{P}(\mathsf{T}^{-1}A) = \mathsf{P}(A)$ for all $A \in \mathcal{F}$; it is said to be ergodic with respect to P if for any measurable set A with $\mathsf{T}^{-1}A = A$ either $\mathsf{P}(A) = 0$ or $\mathsf{P}(A) = 1$ holds; in this case $(X, \mathcal{F}, \mathsf{P}, \mathsf{T})$ is called an ergodic dynamical system. For our purpose we shall consider probability spaces with $X = \mathbb{R}$ and $\mathcal{F}$ as the Borel σ-algebra.

We introduce some further notion to abbreviate the formulation of the main results. Given a domain $\mathcal{D}$, we call a family of analytic functions $\mathcal{L}_1, \ldots, \mathcal{L}_m$ a jointly ergodic universal family with respect to $\mathcal{D}$ if, for any collection of compact subsets $\mathcal{K}_1, \ldots, \mathcal{K}_m$ of $\mathcal{D}$ with connected complements, any family of continuous functions f_j defined on $\mathcal{K}_j$ which is analytic and non-vanishing in the interior of $\mathcal{K}_j$, any ergodic dynamical system $(\mathbb{R}, \mathcal{F}, \mathsf{P}, \mathsf{T})$, almost all real numbers x, and any positive ϵ, there exists a positive integer n such that

$$\max_{1\leq j\leq m} \max_{s\in\mathcal{K}_j} |\mathcal{L}_j(s + i\mathsf{T}^n x) - f_j(s)| < \epsilon \tag{1.2}$$

holds. Here and in the sequel we use the abbreviation $\mathsf{T}x$ for $\mathsf{T}(x)$ and $\mathsf{T}^n x$ is recursively defined by $\mathsf{T} \circ \mathsf{T}^{n-1}x$ and $\mathsf{T}^0 x = x$; the notion *for almost all* x is an abbreviation for *all real numbers except a set of* P*-measure zero.* We call a family of analytic functions $\mathcal{L}_1, \ldots, \mathcal{L}_m$ a strongly jointly ergodic universal family with respect to $\mathcal{D}$ if the assumption on the non-vanishing of the target functions f_j can be dropped.

Theorem 1.1. *Let* $(\mathbb{R}, \mathcal{F}, \mathsf{P}, \mathsf{T})$ *be an ergodic dynamical system where* P *is a probability measure with a positive density function. Then a family of L-functions is jointly (strongly) universal with respect to some domain* $\mathcal{D}$ *if, and only if, it is jointly (strongly) ergodic universal with respect to* $\mathcal{D}$*; in this case,*

$$\liminf_{N\to\infty} \frac{1}{N} \sharp \left\{ n \in \mathbb{N} \cap [1, N] \ : \ \max_{1\leq j\leq m} \max_{s\in\mathcal{K}_j} |\mathcal{L}_j(s + i\mathsf{T}^n x) - f_j(s)| < \epsilon \right\} > 0.$$

It may be noticed that although we do not assume that the approximation problem (1.1) is solvable on a set of τ of positive lower density (as in Voronin's theorem) already a positive proportion of the points of the ergodic orbit fulfill (1.2). This is related to the reccurence property of ergodic orbits due to Poincaré's theorem (see [4]); it also indicates that the question whether (1.1) holds for a set of positive lower density is not directly related to (1.2).

We give an explicit example of an ergodic dynamical system $(\mathbb{R}, \mathcal{F}, \mathsf{P}, \mathsf{T})$ which satisfies the conditions of the theorem. The underlying ergodic transformation on $\mathbb{R}$ has been found by Lind (cf. [4]). Consider the mapping $T : \mathbb{R} \to \mathbb{R}$ given by $\mathsf{T}0 := 0$ and $\mathsf{T}x := \frac{1}{2}(x - \frac{1}{x})$ for $x \neq 0$. It is easy to see that T is an ergodic transformation with respect to the probability measure P defined by

$$\mathsf{P}((\alpha, \beta)) = \frac{1}{\pi} \int_{(\alpha,\beta)} \frac{\mathrm{d}\tau}{1 + \tau^2}; \tag{1.3}$$

for details we refer to [4], Example 2.9.

2. Proof of Theorem 1.1

Assume we are given a domain $\mathcal{D}$ and a jointly universal family of analytic functions $\mathcal{L}_1, \ldots, \mathcal{L}_m$ with respect to $\mathcal{D}$; i.e., for any compact subsets

$\mathcal{K}_1, \ldots, \mathcal{K}_m$ of $\mathcal{D}$ with connected complements, any family of continuous functions f_j defined on $\mathcal{K}_j$, each of which is analytic and non-vanishing in the interior of $\mathcal{K}_j$, and any $\epsilon > 0$, there exists some $\tau > 0$ satisfying

$$\max_{1\leq j\leq m} \max_{s\in\mathcal{K}_j} |\mathcal{L}_j(s+i\tau) - f_j(s)| < \epsilon/2. \tag{2.1}$$

Since T is ergodic, for any positive δ and for almost all real x, there exists some positive integer n such that

$$|\mathsf{T}^n x - \tau| < \delta. \tag{2.2}$$

In order to see this we define the indicator function $1_{\mathcal{A}}$ of the measurable set $\mathcal{A}$ by

$$1_{\mathcal{A}}(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases}$$

Now recall the Birkhoff-Khintchine Ergodic Theorem: *Let* T *be an ergodic transformation on a probability space* $(X, \mathcal{F}, \mathsf{P})$*; if* f *is integrable with respect to* $(X, \mathcal{F}, \mathsf{P})$*, then, for almost all* $x \in X$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} f(\mathsf{T}^n x) = \int_X f \,\mathrm{d}\mathsf{P};$$

i.e., the limit exists and equals the right-hand side (for a proof see [4])). Applying this with the indicator function of the open interval $\mathcal{A} = (\tau - \delta, \tau + \delta)$ yields

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} 1_{\mathcal{A}}(\mathsf{T}^n x) = \int_{\mathbb{R}} 1_{\mathcal{A}} \,\mathrm{d}\mathsf{P} = \mathsf{P}(\mathcal{A}) \tag{2.3}$$

for almost all x. By assumption, P possesses a positive density function, g say; hence,

$$\mathsf{P}(\mathcal{A}) = \int_{\tau-\delta}^{\tau+\delta} g(t) \,\mathrm{d}t$$

is positive, which proves the existence of a positive integer n satisfying (2.2).

Continuous functions on compact sets are in particular uniformly continuous. In our situation the compact set is given by $\mathcal{K}_j + i\tau$. Hence, it follows that

$$\max_{1\leq j\leq m} \max_{s\in\mathcal{K}_j} |\mathcal{L}_j(s+i\mathsf{T}^n x) - \mathcal{L}_j(s+i\tau)| < \epsilon/2$$

if in (2.2) δ is chosen sufficiently small. The latter inequality in combination with (2.1) shows

$$\max_{1\leq j\leq m} \max_{s\in\mathcal{K}_j} |\mathcal{L}_j(s+i\mathsf{T}^n x) - f_j(s)| < \epsilon. \tag{2.4}$$

It remains to prove that this approximation holds on a set of n of positive density. However, this follows already from (2.3) and the positivity of P. More precisely, we note that the ergodic lower density is bounded below by

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} 1_{\mathcal{A}}(\mathsf{T}^n x) \geq \mathsf{P}(\mathcal{A}) > 0,$$

where $\mathcal{A} = \{\tau \in \mathbb{R}_+ : \text{Inequality (2.4) is true}\}$, as follows from inserting this set in place of the open interval in (2.3).

The converse implication is trivial (by setting $\tau = \mathsf{T}^n x$ for some appropriate n). Hence, Theorem 1.1 is proved. □

3. Concluding Remarks

Reviewing the proof of the theorem, it appears that the reasoning for the first implication applies to a family of L-functions which satisfies a discrete joint universality theorem too; the first example for this type of universality was given by Bagchi [1]. The converse implication, however, does not need to be true in general.

Neither Voronin's universality theorem nor any of its generalizations is effective in the sense that an approximating shift τ can be bounded above or below by some non-trivial explicit quantity (apart from a remarkable variation of Garunkštis [2] which applies to a restricted class of target functions). It may be noticed, however, that the expectation that a sequence $\mathsf{T}^n x$ visits a given subset (as, for example, the set $\mathcal{A}$ at the end of the proof of Theorem 1.1) can be computed explicitly from Kac's theorem (which is a quantitative version of Poincaré's recurrence theorem; see [4]).

It is expected that universality itself is an ergodic phenomenon. Unfortunately, our results do not shed any new light on this conjecture. Our reasoning is just based on continuity and basic results from ergodic theory, nevertheless, it leads to a rather astonishing result. We interpret this as another incident for the deepness and beauty of Voronin's universality theorem and its generalizations.

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