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# Diophantine approximations with Fibonacci numbers

par Victoria ZHURAVLEVA

Résumé. Soit $F_n$ le $n$-ième nombre de Fibonacci. Notons $\varphi = \frac{1+\sqrt{5}}{2}$. Nous prouvons les inégalités suivantes pour tous les nombres réels $\alpha$ :
1) $\inf_{n\in\mathbb{N}} ||F_n\alpha|| \le \frac{\varphi-1}{\varphi+2}$,
2) $\liminf_{n\to\infty} ||F_n\alpha|| \le \frac{1}{5}$,
3) $\liminf_{n\to\infty} ||\varphi^n\alpha|| \le \frac{1}{5}$.
Ces résultats sont les meilleurs possibles.

Abstract. Let $F_n$ be the $n$-th Fibonacci number. Put $\varphi = \frac{1+\sqrt{5}}{2}$. We prove that the following inequalities hold for any real $\alpha$:
1) $\inf_{n\in\mathbb{N}} ||F_n\alpha|| \le \frac{\varphi-1}{\varphi+2}$,
2) $\liminf_{n\to\infty} ||F_n\alpha|| \le \frac{1}{5}$,
3) $\liminf_{n\to\infty} ||\varphi^n\alpha|| \le \frac{1}{5}$.
These results are the best possible.

## 1. Introduction

In this paper $||\alpha||$ denotes the distance from a real $\alpha$ to the nearest integer. Let

$$F_1 = F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \dots$$

be the Fibonacci numbers. For convenience we put $F_{-1} = 1$, $F_0 = 0$.

As $F_n$ are distinct integers we deduce from Weyl's Theorem (see [3], Ch.1 §4) that the fractional parts $\{F_n\alpha\}$ are uniformly distributed for almost all real $\alpha$. From the other hand the Fibonacci numbers form a lacunary sequence. So the set

$$\mathcal{N} = \{\alpha : \exists\gamma(\alpha) > 0 \text{ such that } \inf_{n\in\mathbb{N}} ||F_n\alpha|| \ge \gamma(\alpha)\}$$

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is an $\alpha$-winning set for every $\alpha \in (0, 1/2]$ in the sense of Schmidt's $(\alpha, \beta)$-games and hence the Hausdorff dimension of the set $\mathcal{N}$ is equal to one. For the definition and the simplest properties of winning sets see Ch.3 from [4] and [5]. Some quantitative version of Schmidt's results one can find in [1].

Put $\varphi = \frac{1+\sqrt{5}}{2}$. As

$$F_{n+1} = F_n + F_{n-1}, \quad \varphi^{n+1} = \varphi^n + \varphi^{n-1} \tag{1.1}$$

one can easily see that for any real $\alpha$ and for any positive integer $n$ we have the following inequalities:

$$\min_{j=n-1,n,n+1} ||\varphi^j \alpha|| \leq \frac{1}{3}, \quad \min_{j=n-1,n,n+1} ||F_j \alpha|| \leq \frac{1}{3}. \tag{1.2}$$

## 2. A result by A. Dubickas

In [2] Dubickas proved a result related to Diophantine approximations with powers of algebraic numbers. Here we would like to give the formulation of this result.

The length $L(P)$ of a polynomial $P(x) = p_0 + p_1 x + \ldots + p_k x^k \in \mathbb{R}[x]$ is defined as the sum of absolute values of all coefficients of $P(x)$:

$$L(P) = |p_0| + |p_1| + \ldots + |p_k|.$$

The *reduced length* $l(P)$ of a polynomial $P(x)$ is defined as

$$l(P) = \inf_Q L(PQ),$$

where the infinum is taken over all polynomials $Q(x) = q_0 + q_1 x + \ldots + q_r x^r \in \mathbb{R}[x]$ such that $q_0 = 1$ or $q_r = 1$. The reduced length $l(\alpha)$ of an algebraic number $\alpha$ is defined as the reduced length of the irreducible polynomial $P_\alpha(x) \in \mathbb{Z}[x]$ such that $P_\alpha(\alpha) = 0$.

**Theorem 2.1** (A. Dubickas, [2]). *Suppose $\tau > 1$ is an arbitrary algebraic number. Suppose $\alpha$ be a positive real number that lies outside the field $\mathbb{Q}(\tau)$ if $\tau$ is a Pisot or a Salem number. Then it is not possible that all the fractional parts of the form $\{\tau^j \alpha\}$, $j \in \mathbb{N}$ belong to a certain open interval of the length $1/l(\tau)$.*

The history of the question as well as the definitions of Pisot and Salem numbers one can find in [2]. Here we should note that $\varphi = \frac{1+\sqrt{5}}{2}$ is a Pisot number and $l(\varphi) = 1 + \varphi$ (see [2]). In particular Theorem 2.1 shows that under the condition $\alpha \in \mathbb{R}\backslash\mathbb{Q}(\varphi)$ for any $n_0$ there exists an integer $n \geq n_0$ such that

$$\{\varphi^n \alpha\} \notin \Big(\frac{1}{2} - \frac{1}{2(1+\varphi)}, \frac{1}{2} + \frac{1}{2(1+\varphi)}\Big)$$

So Theorem 2.1 leads to the following asymtotical inequality:

$$\liminf_{n\to\infty} ||\varphi^n \alpha|| \leq \frac{1}{2\varphi} = \frac{\sqrt{5}-1}{4}. \tag{2.1}$$

As for the Fibonacci numbers we have the formula

$$F_n = \frac{1}{\sqrt{5}}\Big(\varphi^n - \Big(-\frac{1}{\varphi}\Big)^n\Big)$$

it follows that

$$\liminf_{n\to\infty} ||F_n \alpha|| \leq \frac{1}{2\varphi} = \frac{\sqrt{5}-1}{4} \tag{2.2}$$

in the case $\alpha \in \mathbb{R}\backslash\mathbb{Q}(\varphi)$.

## 3. Statement of results

In this section we summarize all results obtained in this paper.

Let $K$, $N$ be positive integers. Put

$$d_N^K = \max_{\alpha\in\mathbb{R}} \min_{k=K,\dots,K+N-1} ||F_k \alpha||.$$

**Theorem 3.1.**

(1) *We have* $d_1^1 = d_2^1 = \frac{1}{2}$, $d_3^1 = \frac{1}{3}$, $d_4^1 = d_5^1 = \frac{1}{4}$.

(2) *Let* $N \geq 6$, *put* $n = [\frac{N-2}{4}]$. *Then* $d_N^1 = \frac{F_{2n+1}}{F_{2n+2}+F_{2n+4}}$.

**Corollary 3.1.** *We have*

$$\lim_{N\to\infty} d_N^1 = \frac{\varphi - 1}{\varphi + 2}.$$

**Corollary 3.2.** *For any real* $\alpha$ *we have*

$$\inf_{n\in\mathbb{N}} ||F_n \alpha|| \leq \frac{\varphi-1}{\varphi+2}.$$

**Theorem 3.2.** *For* $N \geq 4$, $\alpha_1 = \frac{1}{\varphi+2}$ *we have*

$$\min_{n\leq N} ||F_n \alpha_1|| = \frac{\varphi-1}{\varphi+2}.$$

**Theorem 3.3.** *Let* $\alpha_1 = \frac{1}{\varphi+2}$. *Then* $\forall \varepsilon > 0\ \forall N\ \exists K = K(N)$*:*

$$\min_{k=K,\dots,K+N} ||F_k\alpha_1|| > \frac{1}{5} - \varepsilon.$$

**Theorem 3.4.** *Let* $G_1$ *and* $G_2$ *be arbitrary real numbers, and* $G_n = G_{n-1} + G_{n-2}$ *for* $n \geq 3$. *Then:*

(1) $\max_{G_1,G_2} \min_{n=1,2} ||G_n|| = \frac{1}{2}$,
(2) $\max_{G_1,G_2} \min_{n=1,2,3} ||G_n|| = \frac{1}{3}$,
(3) $\max_{G_1,G_2} \min_{n=1,2,3,4} ||G_n|| = \frac{1}{4}$,
(4) $\max_{G_1,G_2} \min_{n=1,2,3,4,5} ||G_n|| = \frac{1}{4}$,
(5) $\max_{G_1,G_2} \min_{n=1,\dots,k} ||G_n|| = \frac{1}{5}$ *for* $k \geq 6$.

**Corollary 3.3.** *If* $N \geq 6$, *then*

$$\lim_{K\to\infty} d_N^K = \frac{1}{5}.$$

**Corollary 3.4.** *If* $N \geq 6$, *then*

$$\lim_{K\to\infty} \max_{\alpha\in\mathbb{R}} \min_{k=K,\dots,K+N-1} ||\varphi^k\alpha|| = \frac{1}{5}.$$

**Corollary 3.5.** *The following inequalities hold for any real* $\alpha$*:*

$$\liminf_{n\to\infty} ||F_n\alpha|| \leq \frac{1}{5}, \quad \liminf_{n\to\infty} ||\varphi^n\alpha|| \leq \frac{1}{5}.$$

One can see that we improve inequalities (2.1) and (2.2). So we improve Theorem 2.1, but only for $\tau = \varphi$.

## 4. The proof of Theorem 3.1 for $N < 6$

The function $||F_kx||$ is periodic with period smaller than or equal to 1. Also it is symmetric with the respect to the line $x = 1/2$. Thus without loss of generality we consider this function on $[0, 1/2]$.

Since $||F_kx||$ is a piecewise linear function its graph consists of line segments. Let $t$ be an arbitrary integer. Then

$$||F_kx|| = \begin{cases} t - F_kx & \text{if } x \in [\frac{t}{F_k} - \frac{1}{2F_k}, \frac{t}{F_k}], \\ F_kx - t & \text{if } x \in [\frac{t}{F_k}, \frac{t}{F_k} + \frac{1}{2F_k}]. \end{cases}$$

Put $F_N(x) = \min_{k=1,\dots,N} ||F_kx||$.

**Lemma 4.1.** *One can easily see, that for* $x \in [0, \frac{1}{2}]$,

$$F_1(x) = F_2(x) = x,$$

$$F_3(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{3}], \\ 1-2x & \text{if } x \in [\frac{1}{3}, \frac{1}{2}], \end{cases}$$

$$F_4(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{4}], \\ 1-3x & \text{if } x \in [\frac{1}{4}, \frac{1}{3}], \\ 3x-1 & \text{if } x \in [\frac{1}{3}, \frac{2}{5}], \\ 1-2x & \text{if } x \in [\frac{2}{5}, \frac{1}{2}], \end{cases}$$

$$F_5(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{6}], \\ 1-5x & \text{if } x \in [\frac{1}{6}, \frac{1}{5}], \\ 5x-1 & \text{if } x \in [\frac{1}{5}, \frac{1}{4}], \\ 1-3x & \text{if } x \in [\frac{1}{4}, \frac{1}{3}], \\ 3x-1 & \text{if } x \in [\frac{1}{3}, \frac{3}{8}], \\ 2-5x & \text{if } x \in [\frac{3}{8}, \frac{2}{5}], \\ 5x-2 & \text{if } x \in [\frac{2}{5}, \frac{3}{7}], \\ 1-2x & \text{if } x \in [\frac{3}{7}, \frac{1}{2}], \end{cases}$$

$$F_6(x) = \begin{cases} x & \text{if } x \in [0, \frac{1}{9}], \\ 1-8x & \text{if } x \in [\frac{1}{9}, \frac{1}{8}], \\ 8x-1 & \text{if } x \in [\frac{1}{8}, \frac{1}{7}], \\ x & \text{if } x \in [\frac{1}{7}, \frac{1}{6}], \\ 1-5x & \text{if } x \in [\frac{1}{6}, \frac{1}{5}], \\ 5x-1 & \text{if } x \in [\frac{1}{5}, \frac{3}{13}], \\ 2-8x & \text{if } x \in [\frac{3}{13}, \frac{1}{4}], \\ 8x-2 & \text{if } x \in [\frac{1}{4}, \frac{3}{11}], \\ 1-3x & \text{if } x \in [\frac{3}{11}, \frac{1}{3}], \\ 3x-1 & \text{if } x \in [\frac{1}{3}, \frac{4}{11}], \\ 3-8x & \text{if } x \in [\frac{4}{11}, \frac{3}{8}], \\ 8x-3 & \text{if } x \in [\frac{3}{8}, \frac{5}{13}], \\ 2-5x & \text{if } x \in [\frac{5}{13}, \frac{2}{5}], \\ 5x-2 & \text{if } x \in [\frac{2}{5}, \frac{3}{7}], \\ 1-2x & \text{if } x \in [\frac{3}{7}, \frac{1}{2}], \end{cases}$$

$$F_7(x) = \begin{cases} x, & \text{for } x \in [0, \frac{1}{14}], \\ 1-13x, & \text{for } x \in [\frac{1}{14}, \frac{1}{13}], \\ 13x-1, & \text{for } x \in [\frac{1}{13}, \frac{1}{12}], \\ x, & \text{for } x \in [\frac{1}{12}, \frac{1}{9}], \\ 1-8x, & \text{for } x \in [\frac{1}{9}, \frac{1}{8}], \\ 8x-1, & \text{for } x \in [\frac{1}{8}, \frac{1}{7}], \\ 2-13x, & \text{for } x \in [\frac{1}{7}, \frac{2}{13}], \\ 13x-2, & \text{for } x \in [\frac{2}{13}, \frac{1}{6}], \\ 1-5x, & \text{for } x \in [\frac{1}{6}, \frac{1}{5}], \\ 5x-1, & \text{for } x \in [\frac{1}{5}, \frac{2}{9}], \\ 3-13x, & \text{for } x \in [\frac{2}{9}, \frac{3}{13}], \\ 13x-3, & \text{for } x \in [\frac{3}{13}, \frac{4}{17}], \\ 2-8x, & \text{for } x \in [\frac{4}{17}, \frac{1}{4}], \\ 8x-2, & \text{for } x \in [\frac{1}{4}, \frac{3}{11}], \\ 1-3x, & \text{for } x \in [\frac{3}{11}, \frac{3}{10}], \end{cases}$$

$$F_7(x) = \begin{cases} 4-13x & \text{if } x \in [\frac{3}{10}, \frac{4}{13}], \\ 13x-4 & \text{if } x \in [\frac{4}{13}, \frac{5}{16}], \\ 1-3x & \text{if } x \in [\frac{5}{16}, \frac{1}{3}], \\ 3x-1 & \text{if } x \in [\frac{1}{3}, \frac{4}{11}], \\ 3-8x & \text{if } x \in [\frac{4}{11}, \frac{3}{8}], \\ 8x-3 & \text{if } x \in [\frac{3}{8}, \frac{8}{21}], \\ 5-13x & \text{if } x \in [\frac{8}{21}, \frac{5}{13}], \\ 13x-5 & \text{if } x \in [\frac{5}{13}, \frac{7}{18}], \\ 2-5x & \text{if } x \in [\frac{7}{18}, \frac{2}{5}], \\ 5x-2 & \text{if } x \in [\frac{2}{5}, \frac{3}{7}], \\ 1-2x & \text{if } x \in [\frac{3}{7}, \frac{5}{11}], \\ 6-13x & \text{if } x \in [\frac{5}{11}, \frac{6}{13}], \\ 13x-6 & \text{if } x \in [\frac{6}{13}, \frac{7}{15}], \\ 1-2x & \text{if } x \in [\frac{7}{15}, \frac{1}{2}]. \end{cases}$$

Put $d_N = \max_{x\in\mathbb{R}} F_N(x)$. Let $x_N$ be the point where the function $F_N(x)$ attains its maximal value.

From Lemma 4.1 we find $x_N$ and $d_N$ for $N = 1, \ldots, 7$ (see Table 1). So we obtain the first statement of Theorem 3.1.

**Lemma 4.2.** *The graph of the function $F_7(x)$ (see Fig.1) has only one vertex which lies above the line $y = \frac{\varphi-1}{\varphi+2}$. This vertex has coordinates $(\frac{3}{11}, \frac{2}{11})$.*

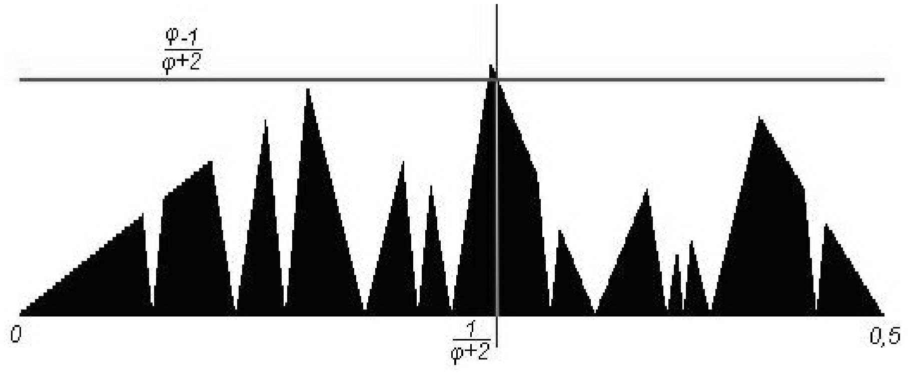



Figure 4.1. $F_7(x)$

We continue to calculate the values of $x_N$ and $d_N$ (see Table 4.1).

Table 4.1.

| $N$ | $x_N$ | $d_N$ |
|---|---|---|
| $1, 2$ | $1/2$ | $1/2$ |
| $3$ | $1/3$ | $1/3$ |
| $4, 5$ | $1/4$ | $1/4$ |
| $6, 7, 8, 9$ | $3/11$ | $2/11$ |
| $10, 11, 12, 13$ | $8/29$ | $5/29$ |
| $14, 15, 16, 17$ | $21/76$ | $13/76$ |
| $18$ | $55/199$ | $34/199$ |

From these results we note that

$$x_6 = \frac{3}{11} = \frac{F_4}{F_4+F_6}, \qquad d_6 = \frac{2}{11} = \frac{F_3}{F_4+F_6},$$
$$x_{10} = \frac{8}{29} = \frac{F_6}{F_6+F_8}, \qquad d_{10} = \frac{5}{29} = \frac{F_5}{F_6+F_8},$$
$$x_{14} = \frac{21}{76} = \frac{F_8}{F_8+F_{10}}, \qquad d_{14} = \frac{13}{76} = \frac{F_7}{F_8+F_{10}},$$
$$x_{18} = \frac{55}{199} = \frac{F_{10}}{F_{10}+F_{12}}, \qquad d_{18} = \frac{34}{199} = \frac{F_9}{F_{10}+F_{12}}.$$

One can see that points $(3/11, 2/11)$, $(8/29, 5/29)$, $(21/76, 13/76)$, $(55/199, 34/199)$ are the elements of the sequence of the points $(a_n, b_n)$ where $a_n = \frac{F_{2n+2}}{F_{2n+2}+F_{2n+4}}$, $b_n = \frac{F_{2n+1}}{F_{2n+2}+F_{2n+4}}$.

We note that $\lim_{n\to\infty} a_n = \frac{1}{\varphi+2}$ and $\lim_{n\to\infty} b_n = \frac{\varphi-1}{\varphi+2}$.

## 5. The nearest integer to $\frac{F_n}{\varphi+2}$

Let $T_n = \sum_{k=1}^{[n/2]} (-1)^{k+1} F_{n-2k} = F_{n-2} - F_{n-4} + F_{n-6} - F_{n-8} + \dots$

**Lemma 5.1.** *$T_n$ is the nearest integer to $\frac{F_n}{\varphi+2}$.*

*Proof.* Consider the sums:

$$\begin{aligned} S_{4t} &= F_0 + F_4 + F_8 + \dots + F_{4t}, \\ S_{4t+1} &= F_1 + F_5 + F_9 + \dots + F_{4t+1}, \\ S_{4t+2} &= F_2 + F_6 + F_{10} + \dots + F_{4t+2}, \\ S_{4t+3} &= F_3 + F_7 + F_{11} + \dots + F_{4t+3}. \end{aligned}$$

From (1.1) one can see that these sums satisfy the following system of linear equations:

$$\begin{cases} S_{4t} + S_{4t+1} + S_{4t+2} + S_{4t+3} = F_{4t+5} - 1, \\ S_{4t} + S_{4t+1} = S_{4t+2}, \\ S_{4t+1} + S_{4t+2} = S_{4t+3}, \\ S_{4t+3} + S_{4t} = S_{4t+1} + F_{4t+3} - F_1. \end{cases}$$

Therefore

$$\begin{aligned} S_{4t} &= \frac{(4F_{4t+3} - F_{4t+5} - 3)}{5}, \\ S_{4t+1} &= \frac{(2F_{4t+5} - 3F_{4t+3} + 1)}{5}, \\ S_{4t+2} &= \frac{(F_{4t+5} + F_{4t+3} - 2)}{5}, \\ S_{4t+3} &= \frac{(3F_{4t+5} - 2F_{4t+3} - 1)}{5}. \end{aligned}$$

Then the explicit formulas for $T_n$ are

$$\begin{aligned} T_{4t} = S_{4(t-1)+2} - S_{4(t-1)} &= \frac{2F_{4t+1} - 3F_{4t-1} + 1}{5}, \\ T_{4t+1} = S_{4(t-1)+3} - S_{4(t-1)+1} &= \frac{F_{4t+1} + F_{4t-1} - 2}{5}, \\ T_{4t+2} = S_{4t} - S_{4(t-1)+2} &= \frac{4F_{4t+3} - F_{4t+5} - F_{4t+1} - F_{4t-1} - 1}{5}, \\ T_{4t+3} = S_{4t+1} - S_{4(t-1)+3} &= \frac{2F_{4t+5} - 3F_{4t+3} - 3F_{4t+1} + 2F_{4t-1} + 2}{5}. \end{aligned}$$

For convenience we rewrite these expressions in the following form:

$$T_{4t} = \frac{2F_{4t} - F_{4t-1} + 1}{5}, \tag{5.1}$$

$$T_{4t+1} = \frac{F_{4t} + 2F_{4t-1} - 2}{5}, \tag{5.2}$$

$$T_{4t+2} = \frac{3F_{4t} + F_{4t-1} - 1}{5}, \tag{5.3}$$

$$T_{4t+3} = \frac{4F_{4t} + 3F_{4t-1} + 2}{5}. \tag{5.4}$$

Now, we calculate the difference $T_n - \frac{F_n}{\varphi+2}$ using Binet's formula.

(1) If $n = 4t$ then

$$T_{4t} - \frac{F_{4t}}{\varphi+2} = \frac{1}{5} + \frac{2\varphi^{4t} - 2(1-\varphi)^{4t} - \varphi^{4t-1} + (1-\varphi)^{4t-1}}{5\sqrt{5}}$$
$$- \frac{\varphi^{4t} - (1-\varphi)^{4t}}{\sqrt{5}(\varphi+2)}$$
$$= \frac{1}{5} + \frac{2\varphi^{4t+1} + 4\varphi^{4t} - \varphi^{4t} - 2\varphi^{4t-1} - 5\varphi^{4t}\varphi(1-\varphi)^{4t-1}}{5\sqrt{5}(\varphi+2)}$$
$$+ \frac{-2\varphi(1-\varphi)^{4t} + 2(1-\varphi)^{4t-1} - 4(1-\varphi)^{4t} + 5(1-\varphi)^{4t}}{5\sqrt{5}(\varphi+2)}$$
$$= \frac{1}{5} + \frac{\varphi^{4t-1}(2\varphi^2 - 2\varphi - 2)}{5\sqrt{5}(\varphi+2)}$$
$$+ \frac{(1-\varphi)^{4t-2}((1-\varphi)^2 + 4(1-\varphi) - 1)}{5\sqrt{5}(\varphi+2)}$$
$$= \frac{1}{5} + \frac{(1-\varphi)(1-\varphi)^{4t-2}}{\sqrt{5}(\varphi+2)}.$$

Similar formulas are obtained in the three remaining cases.

(2) If $n = 4t+1$ then

$$T_{4t+1} - \frac{F_{4t+1}}{\varphi+2} = -\frac{2}{5} + \frac{(2-\varphi)(1-\varphi)^{4t-2}}{\sqrt{5}(\varphi+2)}.$$

(3) If $n = 4t+2$ then

$$T_{4t+2} - \frac{F_{4t+2}}{\varphi+2} = -\frac{1}{5} + \frac{(3-2\varphi)(1-\varphi)^{4t-2}}{\sqrt{5}(\varphi+2)}.$$

(4) If $n = 4t+3$ then

$$T_{4t+3} - \frac{F_{4t+3}}{\varphi+2} = \frac{2}{5} + \frac{(5-3\varphi)(1-\varphi)^{4t-2}}{\sqrt{5}(\varphi+2)}.$$

For convenience we define the function $r(n)$ so that the following equalities hold:

$$T_{4t} - \frac{F_{4t}}{\varphi+2} = \frac{1}{5} + r(4t), \tag{5.5}$$

$$T_{4t+1} - \frac{F_{4t+1}}{\varphi+2} = -\frac{2}{5} + r(4t+1), \tag{5.6}$$

$$T_{4t+2} - \frac{F_{4t+2}}{\varphi+2} = -\frac{1}{5} + r(4t+2), \tag{5.7}$$

$$T_{4t+3} - \frac{F_{4t+3}}{\varphi+2} = \frac{2}{5} + r(4t+3). \tag{5.8}$$

We see that $|r(4t)| > |r(4t+1)| > |r(4t+2)| > |r(4t+3)| > |r(4t+4)|$. So the function $|r(n)|$ is decreasing.

To complete the proof of Lemma 5.1 we need to bound $r(4t)$, $r(4t+1)$, $r(4t+2)$ and $r(4t+3)$ for $t = 1$.

We use the bounds $1.618 < \varphi < 1.619$ and $2.236 < \sqrt{5} < 2.237$. Then

$$-0.030 < \frac{(1-\varphi)(1-\varphi)^2}{\sqrt{5}(\varphi+2)} = r(4) < -0.029, \tag{5.9}$$

$$0.018 < \frac{(2-\varphi)(1-\varphi)^2}{\sqrt{5}(\varphi+2)} = r(5) < 0.019, \tag{5.10}$$

$$-0.012 < \frac{(3-2\varphi)(1-\varphi)^2}{\sqrt{5}(\varphi+2)} = r(6) < -0.011, \tag{5.11}$$

$$0.006 < \frac{(5-3\varphi)(1-\varphi)^2}{\sqrt{5}(\varphi+2)} = r(7) < 0.007. \tag{5.12}$$

So we proved Lemma 5.1 for $t \geq 1$. For $t = 0$ this statement can be verified directly: 0 is the nearest integer to $\frac{F_1}{\varphi+2} = \frac{F_2}{\varphi+2} = \frac{1}{\varphi+2}$, 1 is the nearest integer to $\frac{F_3}{\varphi+2} = \frac{2}{\varphi+2}$. □

## 6. Proof of Theorem 3.1

We fix an arbitrary integer $t \geq 1$.

To prove the second part of Theorem 3.1 we need to find an explicit formula for the function $F_{4t+3}(x)$ for $x \in [\frac{T_{4t+1}}{F_{4t+1}}, \frac{T_{4t}}{F_{4t}}]$.

**Lemma 6.1.** *For $x \in [\frac{T_{4t+1}}{F_{4t+1}}, \frac{T_{4t}}{F_{4t}}]$ we have*

$$(6.1)\quad F_{4t+3}(x) = \begin{cases} F_{4t+1}x - T_{4t+1} & \text{if } x \in [\frac{T_{4t+1}}{F_{4t+1}}, \frac{T_{4t+3}+T_{4t+1}-1}{F_{4t+1}+F_{4t+3}}], \\ T_{4t+3} - 1 - F_{4t+3}x & \text{if } x \in [\frac{T_{4t+3}+T_{4t+1}-1}{F_{4t+1}+F_{4t+3}}, \frac{T_{4t+3}-1}{F_{4t+3}}], \\ F_{4t+3}x - T_{4t+3} + 1 & \text{if } x \in [\frac{T_{4t+3}-1}{F_{4t+3}}, \frac{T_{4t+2}+T_{4t+3}-1}{F_{4t+3}+F_{4t+2}}], \\ T_{4t+2} - F_{4t+2}x & \text{if } x \in [\frac{T_{4t+2}+T_{4t+3}-1}{F_{4t+3}+F_{4t+2}}, \frac{T_{4t+2}}{F_{4t+2}}], \\ F_{4t+2}x - T_{4t+2} & \text{if } x \in [\frac{T_{4t+2}}{F_{4t+2}}, \frac{T_{4t+2}+1}{F_{4t+2}+3}] \\ 1 - 3x & \text{if } x \in [\frac{T_{4t+2}+1}{F_{4t+2}+3}, \frac{T_{4t}-1}{F_{4t}-3}], \\ T_{4t} - F_{4t}x & \text{if } x \in [\frac{T_{4t}-1}{F_{4t}-3}, \frac{T_{4t+3}-T_{4t}}{F_{4t+3}-F_{4t}}], \\ T_{4t+3} - F_{4t+3}x & \text{if } x \in [\frac{T_{4t+3}-T_{4t}}{F_{4t+3}-F_{4t}}, \frac{T_{4t+3}}{F_{4t+3}}], \\ F_{4t+3}x - T_{4t+3} & \text{if } x \in [\frac{T_{4t+3}}{F_{4t+3}}, \frac{T_{4t+3}+T_{4t}}{F_{4t+3}+F_{4t}}], \\ T_{4t} - F_{4t}x & \text{if } x \in [\frac{T_{4t+3}+T_{4t}}{F_{4t+3}+F_{4t}}, \frac{T_{4t}}{F_{4t}}]. \end{cases}$$

*Proof.* We prove Lemma 6.1 by induction. For $t = 1$ the statement follows from Lemma 4.1 (we are interested in the segment $[\frac{1}{5}, \frac{1}{3}]$). We assume that Lemma 6.1 holds for $t = k$.

To make our proof more clear we draw the graph of the function $F_{4k+3}(x)$ on $[\frac{T_{4k+1}}{F_{4k+1}}, \frac{T_{4k}}{F_{4k}}]$ (see Fig.2). The domain below this graph is colored in black. The point $M$ is the intersection of the graphs of the functions $y = 1 - 3x$ and $y = ||F_{4k+2}x||$. The function $F_{4k+3}(x)$ attains its maximal value at this point. The scheme of the graph of the function $F_{4k+7}(x)$ on $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$ is marked with white.

The distance to the nearest integer from $\frac{F_{4k}}{\varphi+2}$, $\frac{F_{4k+1}}{\varphi+2}$, $\frac{F_{4k+2}}{\varphi+2}$, $\frac{F_{4k+3}}{\varphi+2}$ are also marked on the graph (according to (5.5) - (5.8)).

**Remark.** To prove the inductive step we need to compare several numbers. These comparisons can be made in the following way:

1. We have two expressions $A$ and $B$ depending on $T_{4k+7}$, $T_{4k+6}$, $T_{4k+5}$, $T_{4k+4}$, $T_{4k+3}$, $T_{4k+2}$, $F_{4k+7}$, $F_{4k+6}$, $F_{4k+5}$, $F_{4k+4}$, $F_{4k+3}$, $F_{4k+2}$, $F_{4k+1}$, $F_{4k}$. We want to prove that $A > B$. Let us consider $A - B$.
2. The values of $T_i$ are defined in (5.1)-(5.4). We substitute these formulas into $A - B$. The expression obtained depends on $F_{4k+7}$, $F_{4k+6}$, $F_{4k+5}$, $F_{4k+4}$, $F_{4k+3}$, $F_{4k+2}$, $F_{4k+1}$, $F_{4k}$, $F_{4k-1}$.
3. The following formulas can be obtained from the definition of the Fibonacci sequence:

$$\begin{cases} F_{4k+7} = 21F_{4k} + 13F_{4k-1}, \\ F_{4k+6} = 13F_{4k} + 8F_{4k-1}, \\ F_{4k+5} = 8F_{4k} + 5F_{4k-1}, \\ F_{4k+4} = 5F_{4k} + 3F_{4k-1}, \\ F_{4k+3} = 3F_{4k} + 2F_{4k-1}, \\ F_{4k+2} = 2F_{4k} + F_{4k-1}, \\ F_{4k+1} = F_{4k} + F_{4k-1}. \end{cases}$$

We substitute them into the expression obtained in the second step. Now the difference $A - B$ depends on $F_{4k}$ and $F_{4k-1}$.

(4) We use the equality $(-1)^n = F_{n+1}F_{n-1} - F_n^2$ and the fact that $F_{4k} \geq 3$ and $F_{4k-1} \geq 2$ for $k \geq 1$ to obtain the inequality $A - B > 0$.

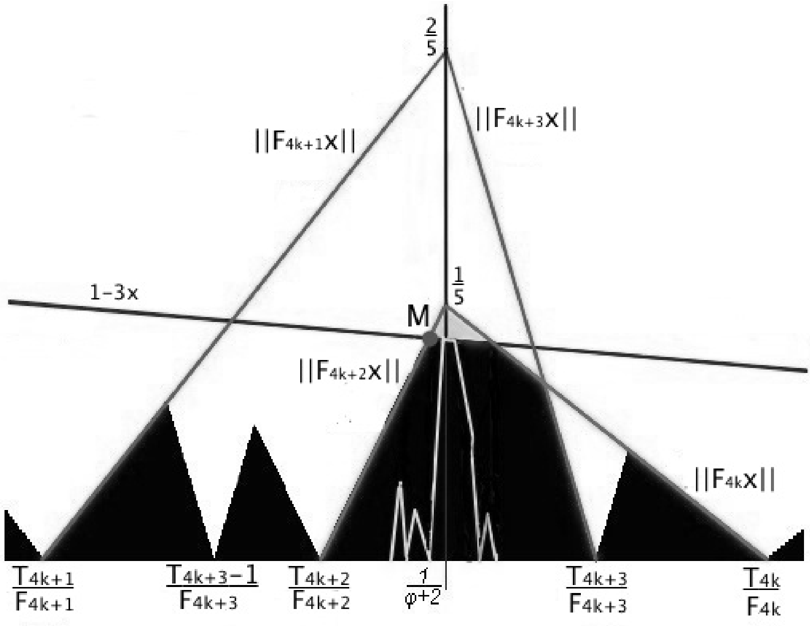



FIGURE 6.1. $F_{4k+3}(x)$

Now we write the proof of one of such inequalities in detail, for example, $\frac{T_{4k+2}}{F_{4k+2}} < \frac{T_{4k+5}}{F_{4k+5}}$.

After substitution described at step 2 we obtain

$$\frac{T_{4k+5}}{F_{4k+5}} - \frac{T_{4k+2}}{F_{4k+2}} = \frac{F_{4k+4} + 2F_{4k+3} - 2}{5F_{4k+5}} - \frac{3F_{4k} + F_{4k-1} - 1}{5F_{4k+2}}.$$

After substitution described at step 3 we obtain

$$\frac{T_{4k+5}}{F_{4k+5}} - \frac{T_{4k+2}}{F_{4k+2}} = \frac{11F_{4k} + 7F_{4k-1} - 2}{40F_{4k} + 25F_{4k-1}} - \frac{3F_{4k} + F_{4k-1} - 1}{10F_{4k} + 5F_{4k-1}}.$$

We rewrite the expression obtained according to step 4:

$$\begin{aligned}\frac{T_{4k+5}}{F_{4k+5}} - \frac{T_{4k+2}}{F_{4k+2}} &= \frac{-10F_{4k}^2 + 10F_{4k-1}^2 + 10F_{4k}F_{4k-1} + 20F_{4k} + 15F_{4k-1}}{(40F_{4k} + 25F_{4k-1})(10F_{4k} + 5F_{4k-1})} \\ &= \frac{-10(F_{4k}F_{4k-2} - F_{4k-1}^2) + 20F_{4k} + 15F_{4k-1}}{(40F_{4k} + 25F_{4k-1})(10F_{4k} + 5F_{4k-1})} \\ &\geq \frac{-10(-1)^{4k-1} + 60 + 30}{(40F_{4k} + 25F_{4k-1})(10F_{4k} + 5F_{4k-1})} > 0.\end{aligned}$$

The procedure described in Remark will be used in our proof several times. Each time we use this procedure we refer to Remark.

We move on to the proof of the inductive step. We assume that on the segment $[\frac{T_{4k+1}}{F_{4k+1}}, \frac{T_{4k}}{F_{4k}}]$ we know the explicit formula for $F_{4k+3}(x)$. We want to find the explicit formula for $F_{4k+7}(x)$ on the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$.

**Proposition 6.1.** *The following inequalities hold:* $\frac{T_{4k+2}}{F_{4k+2}} < \frac{T_{4k+5}}{F_{4k+5}} < \frac{T_{4k+7}-1}{F_{4k+7}} < \frac{T_{4k+6}}{F_{4k+6}} < \frac{1}{\varphi+2} < \frac{T_{4k+7}}{F_{4k+7}} < \frac{T_{4k+4}}{F_{4k+4}} < \frac{T_{4k+3}}{F_{4k+3}}$.

*Proof.* To obtain the inequality $\frac{T_{4k+6}}{F_{4k+6}} < \frac{1}{\varphi+2}$ we should divide (5.7) by $F_{4k+6}$. Similarly, the inequality $\frac{T_{4k+7}}{F_{4k+7}} > \frac{1}{\varphi+2}$ can be obtained from (5.8). The remaining inequalities of the proposition can be obtained by the means of the procedure from Remark. □

**Proposition 6.2.** *The zeros of the function $F_{4k+7}(x)$ on the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$ are the points* $\frac{T_{4k+5}}{F_{4k+5}}$, $\frac{T_{4k+7}-1}{F_{4k+7}}$, $\frac{T_{4k+6}}{F_{4k+6}}$, $\frac{T_{4k+7}}{F_{4k+7}}$, $\frac{T_{4k+4}}{F_{4k+4}}$.

*Proof.* One can easily see that $F_{4k+7}(x) = \min\{F_{4k+3}(x), ||F_{4k+4}x||, ||F_{4k+5}x||, ||F_{4k+6}x||, ||F_{4k+7}x||\}$. Hence we should find the zeros of functions $F_{4k+3}(x)$, $||F_{4k+4}x||$, $||F_{4k+5}x||$, $||F_{4k+6}x||$, $||F_{4k+7}x||$ on the segment considered.

From the inductive assumption it follows that the function $F_{4k+3}(x)$ doesn't have zeros on $(\frac{T_{4k+2}}{F_{4k+2}}, \frac{T_{4k+3}}{F_{4k+3}})$. From Proposition 6.1 we know that $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}] \subset [\frac{T_{4k+2}}{F_{4k+2}}, \frac{T_{4k+3}}{F_{4k+3}}]$. So the function $F_{4k+3}(x)$ doesn't have zeros on the segment considered.

The function $||F_{4k+4}x||$ is equal to zero only when $x = \frac{a}{F_{4k+4}}$, where $a$ is integer. The segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$ has only one point of such a kind which is $x = \frac{T_{4k+4}}{F_{4k+4}}$. We need to compare points $\frac{T_{4k+4}-1}{F_{4k+4}}$ and $\frac{T_{4k+4}+1}{F_{4k+4}}$ with the endpoints of the segment considered to prove that there are no other zeros on $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$.

So we need to prove the inequalities $\frac{T_{4k+4}-1}{F_{4k+4}} < \frac{T_{4k+5}}{F_{4k+5}}$ and $\frac{T_{4k+4}+1}{F_{4k+4}} > \frac{T_{4k+4}}{F_{4k+4}}$. Obviously the inequality $\frac{T_{4k+4}+1}{F_{4k+4}} > \frac{T_{4k+4}}{F_{4k+4}}$ holds. To prove the second one we need to use the procedure from Remark.

A similar argument must be used to the analysis of the three remaining functions.

The function $||F_{4k+5}x||$ is equal to zero on the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$ only when $x = \frac{T_{4k+5}}{F_{4k+5}}$. We need to prove that there are no other zeros of the function $||F_{4k+5}x||$ on the segment considered. Obviously the inequality $\frac{T_{4k+5}}{F_{4k+5}} > \frac{T_{4k+5}-1}{F_{4k+5}}$ holds. The inequality $\frac{T_{4k+5}+1}{F_{4k+5}} > \frac{T_{4k+4}}{F_{4k+4}}$ can be proved by the means of the procedure from Remark.

The function $||F_{4k+6}x||$ is zero only when $x = \frac{T_{4k+6}}{F_{4k+6}}$. From Proposition 6.1 it follows that $\frac{T_{4k+6}}{F_{4k+6}} \in [\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$. We need to prove that $||F_{4k+6}x||$ doesn't have other zeros on the segment considered. The inequalites $\frac{T_{4k+5}}{F_{4k+5}} > \frac{T_{4k+6}-1}{F_{4k+6}}$ and $\frac{T_{4k+6}+1}{F_{4k+6}} > \frac{T_{4k+4}}{F_{4k+4}}$ are proved by the means of the procedure from Remark.

The function $||F_{4k+7}x||$ is zero when $x = \frac{T_{4k+7}-1}{F_{4k+7}}$ and $x = \frac{T_{4k+7}}{F_{4k+7}}$. From Proposition 6.1 we know that these points belong to $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$. We show that the function $||F_{4k+7}x||$ doesn't have other zeros. By the means of the procedure from Remark we prove the inequalities $\frac{T_{4k+5}}{F_{4k+5}} > \frac{T_{4k+7}-2}{F_{4k+7}}$ and $\frac{T_{4k+7}+1}{F_{4k+7}} > \frac{T_{4k+4}}{F_{4k+4}}$.

Proposition 6.2 is proved. □

**Proposition 6.3.** *For each $k$ we have $\frac{T_{4k+6}}{F_{4k+6}} = \frac{T_{4k+2}+1}{F_{4k+2}+3}$ and $\frac{T_{4k+4}}{F_{4k+4}} = \frac{T_{4k}-1}{F_{4k}-3}$. Furthermore,*

$$F_{4k+3}(x) = \begin{cases} F_{4k+2}x - T_{4k+2} & \text{if } x \in [\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+6}}{F_{4k+6}}], \\ 1 - 3x & \text{if } x \in [\frac{T_{4k+6}}{F_{4k+6}}, \frac{T_{4k+4}}{F_{4k+4}}]. \end{cases} \tag{6.2}$$

*Proof.* We prove the equality $\frac{T_{4k+6}}{F_{4k+6}} = \frac{T_{4k+2}+1}{F_{4k+2}+3}$:

$$\begin{aligned} \frac{T_{4k+6}}{F_{4k+6}} - \frac{T_{4k+2}+1}{F_{4k+2}+3} &= \frac{3F_{4k+4} + F_{4k+3} - 1}{5F_{4k+6}} - \frac{3F_{4k} + F_{4k-1} + 4}{5F_{4k+2} + 15} \\ &= \frac{18F_{4k} + 11F_{4k-1} - 1}{65F_{4k} + 40F_{4k-1}} - \frac{3F_{4k} + F_{4k-1} + 4}{10F_{4k} + 5F_{4k-1} + 15} \\ &= \frac{-15(F_{4k}F_{4k-2} - F_{4k-1}^2) - 15}{(65F_{4k} + 40F_{4k-1})(10F_{4k} + 5F_{4k-1} + 15)} = 0. \end{aligned}$$

This equality has the following meaning: the function $F_{4t+3}(x)$ attains its maximal value just at the zero of the function $F_{4t+7}(x)$.

We prove the equality $\frac{T_{4k+4}}{F_{4k+4}} = \frac{T_{4k}-1}{F_{4k}-3}$:

$$\begin{aligned} \frac{T_{4k}-1}{F_{4k}-3} - \frac{T_{4k+4}}{F_{4k+4}} &= \frac{2F_{4k} - F_{4k-1} - 4}{5(F_{4k} - 3)} - \frac{2F_{4k+4} - F_{4k+3} + 1}{5F_{4k+4}} \\ &= \frac{2F_{4k} - F_{4k-1} - 4}{5F_{4k} - 15} - \frac{7F_{4k} + 4F_{4k-1} + 1}{25F_{4k} + 15F_{4k-1}} \\ &= \frac{15(F_{4k}F_{4k-2} - F_{4k-1}^2) + 15}{(5F_{4k} - 15)(25F_{4k} + 15F_{4k-1})} = 0. \end{aligned}$$

These two equalities and the inequality $\frac{T_{4k+2}}{F_{4k+2}} < \frac{T_{4k+5}}{F_{4k+5}}$ (which was proved in Proposition 6.1) lead to formula (6.2). Proposition 6.3 is proved. □

**Proposition 6.4.** *For each $x \in [\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$ we have*

$$||F_{4k+4}x|| = T_{4k+4} - F_{4k+4}x,$$

$$||F_{4k+5}x|| = \begin{cases} F_{4k+5}x - T_{4k+5} & \text{if } x \in [\frac{T_{4k+5}}{F_{4k+5}}, \frac{2T_{4k+5}+1}{2F_{4k+5}}], \\ T_{4k+5} + 1 - F_{4k+5}x & \text{if } x \in [\frac{2T_{4k+5}+1}{2F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}], \end{cases}$$

$$||F_{4k+6}x|| = \begin{cases} T_{4k+6} - F_{4k+6}x & \textit{if } x \in [\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+6}}{F_{4k+6}}], \\ F_{4k+6}x - T_{4k+6} & \textit{if } x \in [\frac{T_{4k+6}}{F_{4k+6}}, \frac{2T_{4k+6}+1}{2F_{4k+6}}], \\ T_{4k+6} + 1 - F_{4k+6}x & \textit{if } x \in [\frac{2T_{4k+6}+1}{2F_{4k+6}}, \frac{T_{4k+4}}{F_{4k+4}}], \end{cases}$$

$$||F_{4k+7}x|| = \begin{cases} T_{4k+7} - 1 - F_{4k+7}x & \textit{if } x \in [\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+7}-1}{F_{4k+7}}], \\ F_{4k+7}x - T_{4k+7} + 1 & \textit{if } x \in [\frac{T_{4k+7}-1}{F_{4k+7}}, \frac{2T_{4k+7}-1}{2F_{4k+7}}], \\ T_{4k+7} - F_{4k+7}x & \textit{if } x \in [\frac{2T_{4k+7}-1}{2F_{4k+7}}, \frac{T_{4k+7}}{F_{4k+7}}], \\ F_{4k+7}x - T_{4k+7} & \textit{if } x \in [\frac{T_{4k+7}}{F_{4k+7}}, \frac{T_{4k+4}}{F_{4k+4}}]. \end{cases}$$

*Proof.* For an explicit formula of the function $||F_nx||$ we should know the zeros of this function as well as the points where this function attains its maximal value.

From Proposition 6.2 we know the zeros of the functions $||F_{4k+4}x||$, $||F_{4k+5}x||$, $||F_{4k+6}x||$, $||F_{4k+7}x||$ on the segment considered. The maximal value of the function $||F_nx||$ is attained at the middle between two neighbouring zeros of this function. So it's enough to prove the following inequalities:

(1) $\frac{T_{4k+5}}{F_{4k+5}} > \frac{2T_{4k+4}-1}{2F_{4k+4}}$ (for the middle of $[\frac{T_{4k+4}-1}{F_{4k+4}}, \frac{T_{4k+4}}{F_{4k+4}}]$)),
(2) $\frac{2T_{4k+5}+1}{2F_{4k+5}} < \frac{T_{4k+4}}{F_{4k+4}}$ (for the middle of $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+5}+1}{F_{4k+5}}]$),
(3) $\frac{2T_{4k+6}-1}{2F_{4k+6}} < \frac{T_{4k+5}}{F_{4k+5}}$ (for the middle of $[\frac{T_{4k+6}-1}{F_{4k+6}}, \frac{T_{4k+6}}{F_{4k+6}}]$),
(4) $\frac{2T_{4k+6}+1}{2F_{4k+6}} < \frac{T_{4k+4}}{F_{4k+4}}$ (for the middle of $[\frac{T_{4k+6}}{F_{4k+6}}, \frac{T_{4k+6}+1}{F_{4k+6}}]$),
(5) $\frac{2T_{4k+7}-3}{2F_{4k+7}} < \frac{T_{4k+5}}{F_{4k+5}}$ (for the middle of $[\frac{T_{4k+7}-2}{F_{4k+7}}, \frac{T_{4k+7}-1}{F_{4k+7}}]$),
(6) $\frac{2T_{4k+7}+1}{2F_{4k+7}} > \frac{T_{4k+4}}{F_{4k+4}}$ (for the middle of $[\frac{T_{4k+7}}{F_{4k+7}}, \frac{T_{4k+7}+1}{F_{4k+7}}]$).

All these inequalities are proved by the means of the procedure from Remark. These inequalities mean that the points $\frac{2T_{4k+5}+1}{2F_{4k+5}}$, $\frac{2T_{4k+6}+1}{2F_{4k+6}}$ belong to the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+6}}{F_{4k+6}}]$. Proposition 6.4 is proved. □

Now we now the explicit formula for the functions $F_{4k+3}(x)$, $||F_{4k+4}x||$, $||F_{4k+5}x||$, $||F_{4k+6}x||$, $||F_{4k+7}x||$ on the segment $[\frac{F_{4k+5}}{T_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$ and the relative position of zeros of these functions on this segment (from Proposition 6.1). So the method of proof of Lemma 6.1 is clear. By the means of the procedure from Remark for comparison of two numbers the formula (6.1) is obtained. □

**Lemma 6.2.** *The graph of the function $F_{4t+3}(x)$ on the segment $[0, \frac{1}{2}]$ has only one vertex which lies above the line $y = \frac{\varphi-1}{\varphi+2}$. This vertex has coordinates $x = \frac{T_{4t+2}+1}{F_{4t+2}+3}$, $y = 1 - 3\frac{T_{4t+2}+1}{F_{4t+2}+3}$. Moreover we have $\frac{T_{4t+2}+1}{F_{4t+2}+3} \in [\frac{T_{4t+2}}{F_{4t+2}}, \frac{T_{4t+3}}{F_{4t+3}}]$.*

*Proof.* We prove Lemma 6.2 by induction. For $t = 1$ the statement is obtained from Lemma 4.2. We assume that Lemma 6.2 holds for $t = k$.

The point $(\frac{T_{4k+2}+1}{F_{4k+2}}, 1 - 3\frac{T_{4k+2}}{F_{4k+2}})$ is the intersection of the lines $y = F_{4k+2}x - T_{4k+2}$ and $y = 1 - 3x$. These lines intersect the line $y = \frac{\varphi-1}{\varphi+2}$ in points $x = \frac{\varphi-1+T_{4k+2}(\varphi+2)}{(\varphi+2)F_{4k+2}}$ and $x = \frac{1}{\varphi+2}$ correspondently. So from the assumption of induction it follows that $F_{4k+3}(x) \geq \frac{\varphi-1}{\varphi+2}$ only if $x \in [\frac{\varphi-1+T_{4k+2}(\varphi+2)}{(\varphi+2)F_{4k+2}}, \frac{1}{\varphi+2}]$.

We prove that the statement of Lemma 6.2 holds for $t = k + 1$.

We note that the points $x = \frac{\varphi-1+T_{4k+2}(\varphi+2)}{(\varphi+2)F_{4k+2}}$ and $x = \frac{1}{\varphi+2}$ belong to the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$. Indeed, for the point $\frac{1}{\varphi+2}$ it follows from Proposition 6.1. For the other one we prove the inequality $\frac{T_{4k+5}}{F_{4k+5}} < \frac{\varphi-1+T_{4k+2}(\varphi+2)}{(\varphi+2)F_{4k+2}}$ by the means of the procedure from Remark and the bound $1.618 < \varphi < 1.619$.

So $F_{4k+3}(x) \leq \frac{\varphi-1}{\varphi+2}$ outside of the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$. Since $F_{4k+7}(x) \leq F_{4k+3}(x)$, then $F_{4k+7}(x) \leq \frac{\varphi-1}{\varphi+2}$ outside of this segment.

From Lemma 6.1 we know the explicit formula for $F_{4k+7}(x)$ on the segment $[\frac{T_{4k+5}}{F_{4k+5}}, \frac{T_{4k+4}}{F_{4k+4}}]$. This formula leads to the fact that the only vertex of the graph is above the line $y = \frac{\varphi-1}{\varphi+2}$. It has the coordinates $(\frac{T_{4k+6}+1}{F_{4k+6}+3}, 1 - 3\frac{T_{4k+2}+1}{F_{4k+2}+3})$.

Lemma 6.2 is proved. □

So we obtain that the maximum of the function $F_{4t+3}(x)$ is equal to $1 - 3\frac{T_{4t+2}+1}{F_{4t+2}+3}$. This function attains its maximum in $x = \frac{T_{4t+2}+1}{F_{4t+2}+3}$.

**Proposition 6.5.** *We have $\frac{T_{4t+2}+1}{F_{4t+2}+3} = \frac{F_{2t+2}}{F_{2t+2}+F_{2t+4}}$ and $1 - 3\frac{T_{4t+2}+1}{F_{4t+2}+3} = \frac{F_{2t+1}}{F_{2t+2}+F_{2t+4}}$.*

*Proof.* To prove the first equality it's enough to show that

$$F_{4t+2} + 3 = F_{2t-1}(F_{2t+2} + F_{2t+4}), \qquad T_{4t+2} + 1 = F_{2t-1}F_{2t+2}.$$

We prove the equality $F_{4t+2} + 3 - F_{2t-1}(F_{2t+2} + F_{2t+4}) = 0$. As for the Fibonacci numbers we have the formula $F_{2t} = F_{t+1}^2 - F_{t-1}^2$ it follows that

$$F_{4t+2}+3-F_{2t-1}(F_{2t+2}+F_{2t+4}) = F_{2t+2}^2-F_{2t}^2+3-F_{2t-1}F_{2t+2}-F_{2t-1}F_{2t+4}.$$

Then we substitute the formulas

$$\begin{cases} F_{2t-1} = F_{2t+1} - F_{2t}, \\ F_{2t+2} = F_{2t+1} + F_{2t}, \\ F_{2t+4} = 3F_{2t+1} + 2F_{2t}. \end{cases}$$

into the expression obtained. This substitution leads to the following equalities:

$$\begin{aligned} F_{2t+2}^2 - F_{2t}^2 + 3 - F_{2t-1}F_{2t+2} - F_{2t-1}F_{2t+4} & \\ &= 3 - 3F_{2t+1}^2 + 3F_{2t}^2 + 3F_{2t+1}F_{2t} \\ &= 3 - 3F_{2t+1}F_{2t-1} + 3F_{2t}^2 \\ &= 3 - 3(-1)^{2t} = 0. \end{aligned}$$

A similar argument must be used to prove the equality $T_{4t+2} + 1 = F_{2t-1}F_{2t+2}$. Since we have $\frac{T_{4t+2}+1}{F_{4t+2}+3} = \frac{F_{2t+2}}{F_{2t+2}+F_{2t+4}}$ then the proof of the remaining equality is easy:

$$\begin{aligned} 1 - 3\frac{T_{4t+2}+1}{F_{4t+2}+3} &= 1 - 3\frac{F_{2t+2}}{F_{2t+2}+F_{2t+4}} \\ &= \frac{F_{2t+2}+F_{2t+4}-3F_{2t+2}}{F_{2t+2}+F_{2t+4}} \\ &= \frac{F_{2t+1}}{F_{2t+2}+F_{2t+4}}. \end{aligned}$$

□

Theorem 3.1 is proved.

## 7. Proofs of Theorem 3.2, Theorem 3.3, Theorem 3.4

*Proof of Theorem 3.2.* From Lemma 4.1, Lemma 6.1 and Proposition 6.1 we have

$$\min_{n=1,\dots,N} ||F_n\alpha_1|| = 1 - 3\alpha_1 = \frac{\varphi - 1}{\varphi - 2}.$$

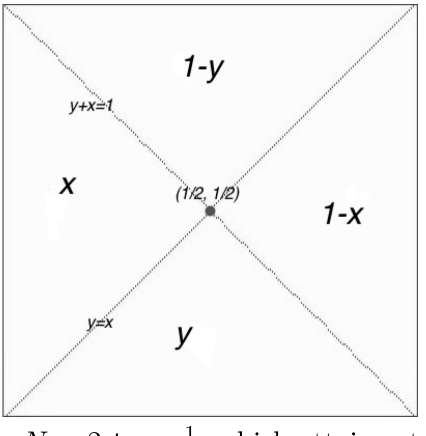
□

*Proof of Theorem 3.3.* From (5.5) - (5.8) we have $||\frac{F_{4t}}{\varphi+2}|| = \frac{1}{5} + O(\varphi^{-4t})$, $||\frac{F_{4t+1}}{\varphi+2}|| = -\frac{2}{5} + O(\varphi^{-4t})$, $||\frac{F_{4t+2}}{\varphi+2}|| = -\frac{1}{5} + O(\varphi^{-4t})$, $||\frac{F_{4t}}{\varphi+2}|| = \frac{2}{5} + O(\varphi^{-4t})$. □

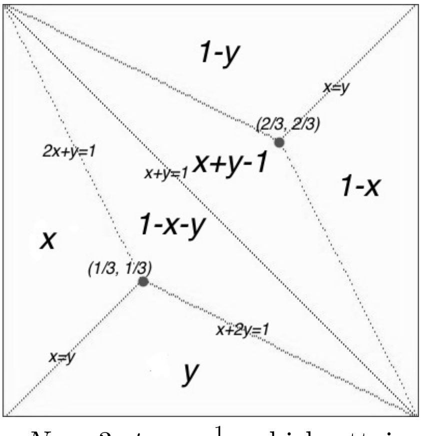
*Proof of Theorem 3.4.* Put $G_1(x,y) = x$, $G_2(x,y) = y$. Then $G_n(x,y) = G_{n-1} + G_{n-2}$ for $n \geq 3$. Let $G^N(x,y) = \min_{n=1,\ldots,N} ||G_n(x,y)||$. Put $t_N = \max_{x,y\in\mathbb{R}} G^N(x,y)$.

For each $N$ we divide the square $[0,1] \times [0,1]$ into domains with the same minimal functions. The maximum is attained at the boundaries of the neighbouring domains.

The graphs below represent such division into domains for $N = 2, \ldots 5$. For the first three graphs the minimal distance to the nearest integer is indicated in each domain. The points where $G^N(x,y)$ attains its maximum, the boundaries of the domains, the lines where $G^N(x,y) = 0$ are also marked.

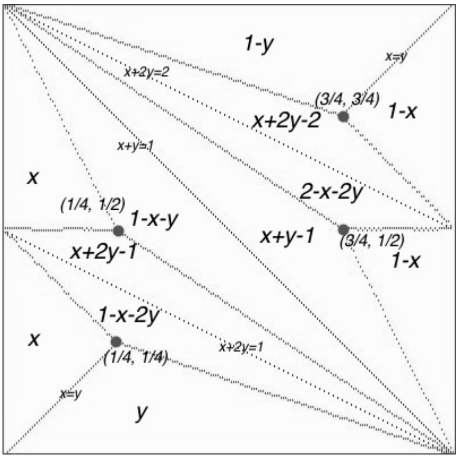



For $N = 2$ $t_N = \frac{1}{2}$, which attains at $x = \frac{1}{2}$, $y = \frac{1}{2}$.




For $N = 3$, $t_N = \frac{1}{3}$, which attains at:

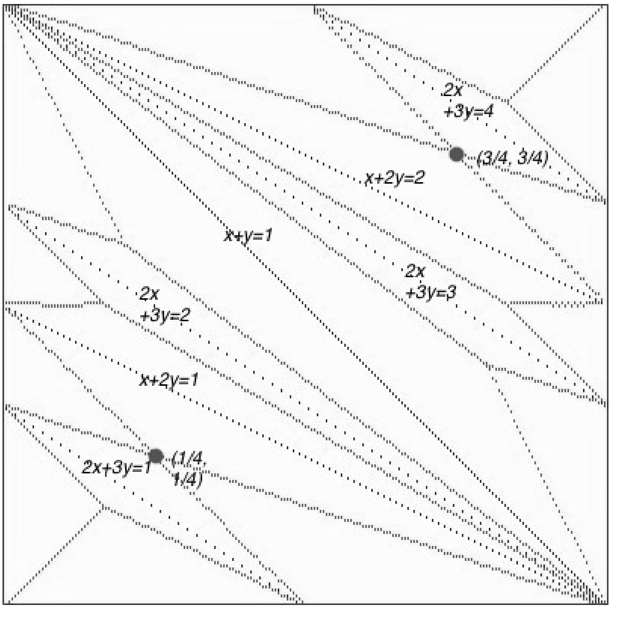
$x_1 = \frac{1}{3}$, $y_1 = \frac{1}{3}$,
$||G_3(x_1, y_1)|| = \frac{2}{3}$,
$x_2 = \frac{2}{3}$, $y_2 = \frac{2}{3}$,
$||G_3(x_2, y_2)|| = \frac{1}{3}$.



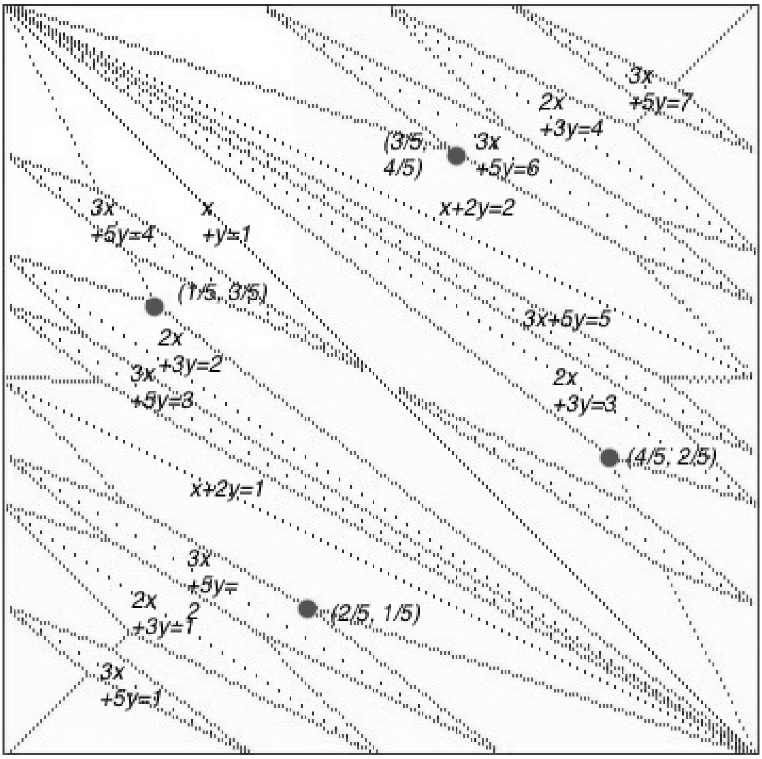

For $N = 4$ $t_N = \frac{1}{4}$, which attains at:

$x_1 = \frac{1}{4}$, $y_1 = \frac{1}{4}$, $||G_3(x_1, y_1)|| = \frac{2}{4}$, $||G_4(x_1, y_1)|| = \frac{3}{4}$,
$x_2 = \frac{1}{4}$, $y_2 = \frac{2}{4}$, $||G_3(x_2, y_2)|| = \frac{3}{4}$, $||G_4(x_2, y_2)|| = \frac{1}{4}$,
$x_3 = \frac{3}{4}$, $y_3 = \frac{3}{4}$, $||G_3(x_3, y_3)|| = \frac{2}{4}$, $||G_4(x_3, y_3)|| = \frac{1}{4}$,
$x_4 = \frac{3}{4}$, $y_4 = \frac{2}{4}$, $||G_3(x_4, y_4)|| = \frac{1}{4}$, $||G_4(x_4, y_4)|| = \frac{3}{4}$.

On the next two graphs we only mark the lines where $G^N(x, y)$ is equal to zero and the points where this function attains its maximal value.



For $N = 5$ $t_N = \frac{1}{4}$, which attains at:
$x_1 = \frac{1}{4}$, $y_1 = \frac{1}{4}$, $||G_3(x_1, y_1)|| = \frac{2}{4}$, $||G_4(x_1, y_1)|| = \frac{3}{4}$, $||G_5(x_1, y_1)|| = \frac{1}{4}$
$x_2 = \frac{3}{4}$, $y_2 = \frac{3}{4}$, $||G_3(x_2, y_2)|| = \frac{2}{4}$, $||G_4(x_2, y_2)|| = \frac{1}{4}$, $||G_5(x_2, y_2)|| = \frac{3}{4}$




For $N = 6$ $t_N = \frac{1}{5}$, which attains at:
$x_1 = \frac{2}{5}$, $y_1 = \frac{1}{5}$, $||G_3(x_1, y_1)|| = \frac{3}{5}$, $||G_4(x_1, y_1)|| = \frac{4}{5}$, $||G_5(x_1, y_1)|| = \frac{2}{5}$, $||G_6(x_1, y_1)|| = \frac{1}{5}$
$x_2 = \frac{1}{5}$, $y_2 = \frac{3}{5}$, $||G_3(x_2, y_2)|| = \frac{4}{5}$, $||G_4(x_2, y_2)|| = \frac{2}{5}$, $||G_5(x_2, y_2)|| = \frac{1}{5}$, $||G_6(x_2, y_2)|| = \frac{3}{5}$
$x_3 = \frac{3}{5}$, $y_3 = \frac{4}{5}$, $||G_3(x_3, y_3)|| = \frac{2}{5}$, $||G_4(x_3, y_3)|| = \frac{1}{5}$, $||G_5(x_3, y_3)|| = \frac{3}{5}$, $||G_6(x_3, y_3)|| = \frac{4}{5}$
$x_4 = \frac{4}{5}$, $y_4 = \frac{2}{5}$, $||G_3(x_4, y_4)|| = \frac{1}{5}$, $||G_4(x_4, y_4)|| = \frac{3}{5}$, $||G_5(x_4, y_4)|| = \frac{4}{5}$, $||G_6(x_4, y_4)|| = \frac{2}{5}$

We note that each of these sequences is periodic. For example, for $x = \frac{2}{5}$ and $y = \frac{1}{5}$ the sequence $G_N(x, y)$ is as follows: $\frac{2}{5}$, $\frac{1}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{2}{5}$, $\frac{1}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{2}{5}$, $\frac{1}{5}$ .... It means that $||G^N(\frac{2}{5}, \frac{1}{5})|| = \frac{1}{5}$ for $N \geq 2$.

For each $N$ we have the inequality $G^{N+1}(x,y) \leq G^N(x,y)$. Hence, $t_N = \frac{1}{5}$ for $N \geq 6$ . □

## References


[1] R. K. Akhunzhanov, *On the distribution modulo 1 of exponential sequences*. Mathematical notes **76:2** (2004), 153–160.

[2] A. Dubickas, *Arithmetical properties of powers of algebraic numbers*. Bull. London Math. Soc. **38** (2006), 70–80.

[3] L. Kuipers, H. Niederreiter, *Uniform distribution of sequences*. John Wiley & Sons, 1974.

[4] W. M. Schmidt, *Diophantine approximations*. Lect. Not. Math. **785**, 1980.

[5] W. M. Schmidt, *On badly approximable numbers and certain games*. Trans. Amer. Math. Soc. **623** (1966), 178–199.



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