JOURNAL
de Théorie des Nombres
de BORDEAUX
anciennement Séminaire de Théorie des Nombres de Bordeaux

Yuichi KAMIYA et Leo MURATA

Certain codes related to generalized paperfolding sequences

Tome 27, n° 1 (2015), p. 149-169.

<http://jtnb.cedram.org/item?id=JTNB_2015__27_1_149_0>

Journal de Théorie des Nombres de Bordeaux **27** (2015), 149–169

Certain codes related to generalized paperfolding sequences

par Yuichi KAMIYA et Leo MURATA

Résumé. Soit RBC le code binaire réfléchi, qui est aussi connu sous le nom de code de Gray, soit S_{RBC} la somme des chiffres pour RBC, et soit $\{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty}$ la suite du pliage régulier de papier. Les auteurs ont montré que la différence première de la somme des chiffres pour RBC, $\{S_{\text{RBC}}(n) - S_{\text{RBC}}(n-1)\}_{n=1}^{\infty}$, coïncide avec $\{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty}$. Pour toute suite infinie $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$, avec $b_k \in \{-1, 1\}$, on peut construire une suite infinie $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$, appelée suite de pliage de papier généralisée associée à $\mathbf{b}$. Dans cet article, supposons que la suite $\mathbf{b}$ est periodique, nous proposons un nouveau code (de numération) $\mathcal{C}_{\mathbf{b}}$ défini par $\mathbf{b}$, et nous étudions les propriétés du code $\mathcal{C}_{\mathbf{b}}$ dans le Théorème 1.2. Nous montrons que la différence première de la somme des chiffres pour $\mathcal{C}_{\mathbf{b}}$, $\{S_{\mathcal{C}_{\mathbf{b}}}(n) - S_{\mathcal{C}_{\mathbf{b}}}(n-1)\}_{n=1}^{\infty}$, coïncide avec la suite de pliage de papier généralisée $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$ (Théorème 1.1). Puis nous donnons une formule exacte pour la moyenne de la somme des chiffres pour $\mathcal{C}_{\mathbf{b}}$ dans le Théorème 1.3.

Abstract. Let RBC be the reflected binary code, which is also called the Gray code, S_{RBC} be the sum of digits function for RBC, and $\{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty}$ be the regular paperfolding sequence. In their previous work the authors proved that the difference function of the sum of digits function for RBC, $\{S_{\text{RBC}}(n) - S_{\text{RBC}}(n-1)\}_{n=1}^{\infty}$, coincides with $\{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty}$. From an infinite sequence $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ with $b_k \in \{-1, 1\}$, one can construct an infinite sequence $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$ which is called the generalized paperfolding sequence with respect to $\mathbf{b}$. In this paper, when we assume $\mathbf{b}$ is periodic, we propose a new numeration code $\mathcal{C}_{\mathbf{b}}$, and study some properties of the code $\mathcal{C}_{\mathbf{b}}$ in Theorem 1.2. We can prove that the difference function of the sum of digits function $S_{\mathcal{C}_{\mathbf{b}}}$ for $\mathcal{C}_{\mathbf{b}}$, $\{S_{\mathcal{C}_{\mathbf{b}}}(n) - S_{\mathcal{C}_{\mathbf{b}}}(n-1)\}_{n=1}^{\infty}$, coincides with the generalized paperfolding sequence $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$ (Theorem 1.1). We also give an exact formula for the average of $S_{\mathcal{C}_{\mathbf{b}}}$ in Theorem 1.3.

Manuscrit reçu le 17 août 2012, révisé le 24 février 2013, accepté le 19 mars 2013.
Mots clefs. Paperfolding sequence, Numeration system, Sum of digits function.
Mathematics Subject Classification. 11B85, 11A25.

1. Introduction

We start from an interesting relation between the reflected binary code and the paperfolding sequence.

The reflected binary code (RBC) is one of the Gray codes. We show how to construct RBC in Example 2.1 of Section 2. Here we compare, for $0 \le n \le 7$, its binary code (BC) and RBC:

n	BC	RBC
0	0	0
1	1	1
2	10	11
3	11	10
4	100	110
5	101	111
6	110	101
7	111	100

For the code

$$\{\mathrm{RBC}(n)\}_{n=0}^{\infty} = \{0, 1, 11, 10, 110, 111, 101, 100, \\ 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000, \ldots\},$$

we introduce two important functions, the sum of digits function $S_{\mathrm{RBC}}(n)$ and its difference function $H_{\mathrm{RBC}}(n) = S_{\mathrm{RBC}}(n) - S_{\mathrm{RBC}}(n-1)$. The initial 16 values of these functions are

$$\{S_{\mathrm{RBC}}(n)\}_{n=0}^{\infty} = \{0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 3, 2, 1, \ldots\},$$

$$\{H_{\mathrm{RBC}}(n)\}_{n=1}^{\infty} = \{1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, 1, \ldots\}.$$

Let us fold a paper repeatedly always in the same way and open the paper. Then we get a sequence of 2 types of folds, '$\wedge$' and '$\vee$'. We assign 1 for $\wedge$ and -1 for $\vee$, then we obtain the so-called *regular paperfolding sequence* $\{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty}$. The initial 16 values are

(1.1)

$$\{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty} = \{1, 1, -1, 1, 1, -1, -1, 1, 1, 1, -1, -1, 1, -1, -1, 1, \ldots\}.$$

We can easily notice that

$$\{H_{\mathrm{RBC}}(n)\}_{n=1}^{\infty} = \{P_{\mathbf{b}_0}(n)\}_{n=1}^{\infty}, \tag{1.2}$$

and this is already proved in [3].

The regular paperfolding sequence is the simplest example of *generalized paperfolding sequences*. Then we want to study, for a given generalized paperfolding sequence $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$, whether we can find out a new code $\mathcal{C}_{\mathbf{b}}$, which satisfies a relation similar to (1.2), that is, the sequence $\{S_{\mathcal{C}_{\mathbf{b}}}(n) - S_{\mathcal{C}_{\mathbf{b}}}(n-1)\}_{n=1}^{\infty}$ coincides with $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$, where $S_{\mathcal{C}_{\mathbf{b}}}$ is the sum of digits function for the code $\mathcal{C}_{\mathbf{b}}$.

We give an affirmative answer to this question (Theorem 1.1), and this is the first purpose of this paper.

Here we introduce generalized paperfolding sequences precisely. Let $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ be a sequence with $b_k \in \{-1, 1\}$ and $b_0 = 1$. By making use of $\mathbf{b}$, let us define the words $w_1, w_2, \ldots, w_j, \ldots$, inductively; $w_1 = b_0$ and $w_j = w_{j-1} \cdot b_{j-1} \cdot (-w_{j-1})^R$, $j \geq 2$, where $\cdot$ means concatenation of words, and for $w = a_1 a_2 \cdots a_n$, $(-w)^R$ means $(-w)^R = (-a_n) \cdots (-a_2)(-a_1)$. When j tends to infinity, one can get an infinite sequence $\lim_{j\to\infty} w_j = \{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$ which is called the *paperfolding sequence with respect to* $\mathbf{b}$. When we take the simplest sequence $\mathbf{b}_0 = \{1, 1, 1, 1, \ldots\}$, then $w_1 = 1$, $w_2 = 11{-1}$, $w_3 = 11{-1}11{-1}{-1}{-1}$, and so on, and get (1.1). As for paperfolding sequences, we can refer to Allouche and Shallit [1], Section 6.5.

From now on, throughout this paper, we assume that the sequence $\mathbf{b}$ is periodic.

Definition 1.1. Let $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ be a periodic sequence with $b_k \in \{-1, 1\}$ and $b_0 = 1$, and K be the minimal period of $\mathbf{b}$. For $0 \leq r \leq 2^K - 1$, let us express the rth word of RBC as $\mathrm{RBC}(r) = s_{K-1} \cdots s_2 s_1 s_0$, where s_{K-1} may be 0. Then the map $\eta : \{0, 1, 2, \cdots, 2^K - 1\} \to \{-1, 0, 1\}^K$ is defined by

$$\eta(r) = (b_{K-1}s_{K-1}) \cdots (b_2 s_2) \cdot (b_1 s_1) \cdot (b_0 s_0).$$

Namely,

$\eta(r)$	word
$\eta(0)$	$0 \cdots 0\,0\;0\;0$
$\eta(1)$	$0 \cdots 0\,0\;0\;b_0$
$\eta(2)$	$0 \cdots 0\,0\;b_1 b_0$
$\eta(3)$	$0 \cdots 0\,0\;b_1\,0$
$\eta(4)$	$0 \cdots 0\,b_2 b_1\,0$
$\eta(5)$	$0 \cdots 0\,b_2 b_1 b_0$
$\eta(6)$	$0 \cdots 0\,b_2\,0\;b_0$
$\eta(7)$	$0 \cdots 0\,b_2\,0\;0$
$\vdots$	$\vdots$
$\eta(2^K - 1)$	$b_{K-1} 0 \cdots\cdots 0$

The map η is obviously injective.

We denote the set of positive integers by $\mathbf{N}$. The integer and complex numbers are denoted by $\mathbf{Z}$ and $\mathbf{C}$, respectively.

Here we introduce codes related to generalized paperfolding sequences.

Definition 1.2. Let $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ be a periodic sequence with $b_k \in \{-1, 1\}$ and $b_0 = 1$, K be the minimal period of $\mathbf{b}$, and σ be the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & \cdots & 2^K - 2 & 2^K - 1 \\ 2^K - 1 & 2^K - 2 & \cdots & 1 & 0 \end{pmatrix}.$$

For $n \in \mathbf{N} \cup \{0\}$, let m and r be the integers such that $n = 2^K m + r$, $0 \leq r \leq 2^K - 1$.

Then the *code* $\mathcal{C}_{\mathbf{b}}$ *induced by* $\mathbf{b}$ is defined by

$$\mathcal{C}_{\mathbf{b}}(n) = \begin{cases} \eta(n), & \text{if } 0 \le n \le 2^K - 1, \\ \mathcal{C}_{\mathbf{b}}(m) \cdot \eta(\sigma^m(r)), & \text{if } n \ge 2^K. \end{cases}$$

In Section 2, we present a few examples of $\mathcal{C}_{\mathbf{b}}$.

It should be noted that the code $\mathcal{C}_{\mathbf{b}}$ can be realized in a different way. We take the basic sequence $\{0,1,1,0\}$ and repeat this infinitely many times:

$$\mathbf{U}_0 = \{u_0(n)\}_{n=0}^{\infty} = \{0,1,1,0,0,1,1,0,0,1,1,0,0,1,1,0,\ldots\}.$$

This sequence is the first digits of RBC. We prolong the basic sequence two times as $\{0,0,1,1,1,1,0,0\}$, repeat this infinitely many times, and multiply by b_1. Then we get the sequence

$$\mathbf{U}_1 = \{u_1(n)\}_{n=0}^{\infty} = \{0,0,b_1,b_1,b_1,b_1,0,0,0,0,b_1,b_1,b_1,b_1,0,0,\ldots\},$$

and similarly $\mathbf{U}_2 = \{u_2(n)\}_{n=0}^{\infty}$, $\mathbf{U}_3 = \{u_3(n)\}_{n=0}^{\infty}$, and so on. Then we have the code word for n, namely,

$$\mathcal{C}_{\mathbf{b}}(n) = \cdots u_J(n) \cdots u_2(n) \cdot u_1(n) \cdot u_0(n).$$

For the code $\mathcal{C}_{\mathbf{b}}$, we define the sum of digits function $S_{\mathcal{C}_{\mathbf{b}}}$ by

$$S_{\mathcal{C}_{\mathbf{b}}}(n) = \text{the sum of digits of } \mathcal{C}_{\mathbf{b}}(n),$$

and its difference function $H_{\mathcal{C}_{\mathbf{b}}} : \mathbf{N} \to \mathbf{Z}$ by

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = S_{\mathcal{C}_{\mathbf{b}}}(n) - S_{\mathcal{C}_{\mathbf{b}}}(n-1). \tag{1.3}$$

Then we can prove the coincidence.

Theorem 1.1. *Let* $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ *be a periodic sequence with* $b_k \in \{-1,1\}$ *and* $b_0 = 1$*. Let* $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$ *be the paperfolding sequence with respect to* $\mathbf{b}$*, and* $H_{\mathcal{C}_{\mathbf{b}}}$ *be the difference function* (1.3)*. Then we have* $H_{\mathcal{C}_{\mathbf{b}}} = P_{\mathbf{b}}$*.*

When the sequence $\mathbf{b}$ is periodic, it is known that $P_{\mathbf{b}}$ is a 2-automatic sequence (see, e.g., Allouche and Shallit [1], Theorem 6.5.4). Hence $H_{\mathcal{C}_{\mathbf{b}}}$ is also 2-automatic. This implies that $S_{\mathcal{C}_{\mathbf{b}}}$ is a 2-regular sequence (see, [1], Theorems 16.1.5 and 16.4.1).

For $\mathrm{RBC}(n) = s_J \cdots s_2 s_1 s_0$, let $D_{\mathrm{RBC}}(n) = \sum_{j=0}^{J} s_j 2^j$. Then the function D_{RBC} is a bijection from $\mathbf{N} \cup \{0\}$ to $\mathbf{N} \cup \{0\}$.

Here we consider the similar function $D_{\mathcal{C}_{\mathbf{b}}}$ for the code $\mathcal{C}_{\mathbf{b}}$. Namely, for $\mathcal{C}_{\mathbf{b}}(n) = u_J \cdots u_2 u_1 u_0$, let $D_{\mathcal{C}_{\mathbf{b}}}(n) = \sum_{j=0}^{J} u_j 2^j$. Then we have

Theorem 1.2. *Let* $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ *be a periodic sequence with* $b_k \in \{-1,1\}$ *and* $b_0 = 1$*,* K *be the minimal period of* $\mathbf{b}$*, and* $\mathcal{C}_{\mathbf{b}}$ *be the code induced by* $\mathbf{b}$*. Let* $K \ge 2$*. Then the function* $D_{\mathcal{C}_{\mathbf{b}}}$ *is a bijection from* $\mathbf{N} \cup \{0\}$ *to* $\mathbf{Z}$*.*

This result shows that $\mathcal{C}_{\mathbf{b}}$ is natural as a numeration system for $\mathbf{Z}$ (see Section 4).

In the previous paper [3], the present authors proved the existence of a bijective map between arithmetical functions (Theorem 1.1, [3]). We can generalize it as follows: Let $q \geq 2$ be an integer. Let $\mathcal{A}$ be the set of all arithmetical functions $g : \mathbf{N} \cup \{0\} \to \mathbf{C}$ with $g(0) = 0$. For $g \in \mathcal{A}$, we define the map $\Phi_q : \mathcal{A} \to \mathcal{A}$ by

$$(\Phi_q(g))(n) = \sum_{k=0}^{\infty} \sum_{0 \leq a \leq \frac{n}{q^k}} g(a),$$

and for $S \in \mathcal{A}$, we define the map $\Psi_q : \mathcal{A} \to \mathcal{A}$ by

(1.4)

$$(\Psi_q(S))(n) = \begin{cases} 0, & \text{if } n = 0, \\ S(n) - S(n-1) - \Big(S\big(\frac{n}{q}\big) - S\big(\frac{n}{q} - 1\big)\Big), & \text{if } n \geq q \text{ and } n \equiv 0 \pmod{q}, \\ S(n) - S(n-1), & \text{if } n \not\equiv 0 \pmod{q}. \end{cases}$$

Then *the map* Φ_q *is bijective with the inverse map* $\Phi_q^{-1} = \Psi_q$.

We introduced the new code $\mathcal{C}_{\mathbf{b}}$ in the above, and clearly $S_{\mathcal{C}_{\mathbf{b}}} \in \mathcal{A}$. Then we can apply the argument in [3] and derive the Delange type result (cf. Delange [2]) for the average of sum of digits function $S_{\mathcal{C}_{\mathbf{b}}}$.

Theorem 1.3. *Let* $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ *be a periodic sequence with* $b_k \in \{-1, 1\}$ *and* $b_0 = 1$, K *be the minimal period of* $\mathbf{b}$, $\mathcal{C}_{\mathbf{b}}$ *be the code induced by* $\mathbf{b}$, *and* $S_{\mathcal{C}_{\mathbf{b}}}$ *be the sum of digits function for* $\mathcal{C}_{\mathbf{b}}$. *Let* $f = \Psi_{2^K}(S_{\mathcal{C}_{\mathbf{b}}})$ *and* $\xi(x) = \sum_{0 \leq n \leq x} f(n)$. *Then, for any positive integer* N,

$$\frac{1}{N} \sum_{n=0}^{N-1} S_{\mathcal{C}_{\mathbf{b}}}(n) = \frac{\log N}{2 \log 2^K} \sum_{k=0}^{K-1} b_k + F\Big(\frac{\log N}{\log 2^K}\Big),$$

where the function $F(x)$ *is defined by either of the following two ways* (I) *and* (II), $F(x)$ *is periodic with period* 1, *continuous, and nowhere differentiable*:

(I)

$$F(x) = \frac{1 + \lfloor x \rfloor - x}{2} \sum_{k=0}^{K-1} b_k + 2^{K(\lfloor x \rfloor - x)} \sum_{r=0}^{\infty} \int_0^{2^{K(x - \lfloor x \rfloor)}} \Big(\xi(2^{Kr} t) - \frac{1}{2} \sum_{k=0}^{K-1} b_k\Big) dt,$$

(II) $F(x) = \sum_{k \in \mathbf{Z}} D_k e^{2\pi i k x}$ *whose Fourier coefficients are given by*

$$\begin{cases} D_0 = \Big(\dfrac{1}{2} - \dfrac{1}{\log 2^K}\Big)L(0,f) + \dfrac{L'(0,f)}{\log 2^K}, \\ D_k = \dfrac{L(\frac{2\pi ik}{\log 2^K}, f)}{2\pi ik(\frac{2\pi ik}{\log 2^K}+1)}, \qquad k \neq 0, \end{cases}$$

where $L(s,f)$ is the Dirichlet L-function with coefficients $f(n)$, and $L'(s,f)$ is its derivative (*see Section* 5)*.*

In [3] the authors mainly discuss about sum of digits functions for Gray codes. The code $\mathcal{C}_{\mathbf{b}}$ is usually not a Gray code. However, since $\Psi_{2^K}(S_{\mathcal{C}_{\mathbf{b}}})$ satisfies Lemma 5.1 in Section 5 which is corresponding to Theorem 1.2 of [3], we can obtain a Delange type result for $\mathcal{C}_{\mathbf{b}}$. This means that Delange type results hold for rather wide family of codes.

Acknowledgement: This paper is a fruit of discussion between J.-P. Allouche and the authors. The authors would like to express their hearty gratitude to J.-P. Allouche. The authors also would like to express their sincere gratitude to the referee for his/her valuable comments.

2. Examples of $\mathcal{C}_{\mathbf{b}}$

In this section we mention a few examples of $\mathcal{C}_{\mathbf{b}}$.

Example 2.1. (The reflected binary code (RBC))
We take $\mathbf{b} = \{1,1,1,1,\dots\}$. Then $K=1$,

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$\eta(r)$	word
$\eta(0)$	0
$\eta(1)$	1

and

n	m	r	$\mathcal{C}_{\mathbf{b}}(m)\cdot\eta(\sigma^m(r))$	$\mathcal{C}_{\mathbf{b}}$	$S_{\mathcal{C}_{\mathbf{b}}}$	$H_{\mathcal{C}_{\mathbf{b}}}$	$D_{\mathcal{C}_{\mathbf{b}}}$
0	0	0	$\eta(0)$	0	0		0
1	0	1	$\eta(1)$	1	1	1	1
2	1	0	$\mathcal{C}_{\mathbf{b}}(1)\cdot\eta(\sigma^1(0))$	11	2	1	3
3	1	1	$\mathcal{C}_{\mathbf{b}}(1)\cdot\eta(\sigma^1(1))$	10	1	-1	2
4	2	0	$\mathcal{C}_{\mathbf{b}}(2)\cdot\eta(\sigma^0(0))$	110	2	1	6
5	2	1	$\mathcal{C}_{\mathbf{b}}(2)\cdot\eta(\sigma^0(1))$	111	3	1	7
6	3	0	$\mathcal{C}_{\mathbf{b}}(3)\cdot\eta(\sigma^1(0))$	101	2	-1	5
7	3	1	$\mathcal{C}_{\mathbf{b}}(3)\cdot\eta(\sigma^1(1))$	100	1	-1	4
8	4	0	$\mathcal{C}_{\mathbf{b}}(4)\cdot\eta(\sigma^0(0))$	1100	2	1	12
9	4	1	$\mathcal{C}_{\mathbf{b}}(4)\cdot\eta(\sigma^0(1))$	1101	3	1	13
10	5	0	$\mathcal{C}_{\mathbf{b}}(5)\cdot\eta(\sigma^1(0))$	1111	4	1	15
11	5	1	$\mathcal{C}_{\mathbf{b}}(5)\cdot\eta(\sigma^1(1))$	1110	3	-1	14
12	6	0	$\mathcal{C}_{\mathbf{b}}(6)\cdot\eta(\sigma^0(0))$	1010	2	-1	10
13	6	1	$\mathcal{C}_{\mathbf{b}}(6)\cdot\eta(\sigma^0(1))$	1011	3	1	11
14	7	0	$\mathcal{C}_{\mathbf{b}}(7)\cdot\eta(\sigma^1(0))$	1001	2	-1	9
15	7	1	$\mathcal{C}_{\mathbf{b}}(7)\cdot\eta(\sigma^1(1))$	1000	1	-1	8

Remark 1. The reflected binary code (RBC) is the code $\mathcal{C}_{\mathbf{b}}$ with the same $\mathbf{b}$, σ, and η as in Example 2.1. It is worth to show the relation between the

base-2 representation $n = r_J \cdots r_2 r_1 r_0$ and $\mathrm{RBC}(n) = s_J \cdots s_2 s_1 s_0$. By the later Lemma 4.1, $\mathrm{RBC}(n)$ is expressed as

$$\mathrm{RBC}(n) = r_J \cdot \sigma^{r_J}(r_{J-1}) \cdots \sigma^{r_2}(r_1) \cdot \sigma^{r_1}(r_0).$$

Hence the digits s_j of $\mathrm{RBC}(n)$ satisfy the relation

$$s_j = \sigma^{r_{j+1}}(r_j), \quad 0 \le j \le J,$$

where r_{J+1} is defined to be 0.

Example 2.2. We take $\mathbf{b} = \{1, -1, 1, -1, \ldots\}$. Then $K = 2$,

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

$\eta(r)$	word
$\eta(0)$	00
$\eta(1)$	01
$\eta(2)$	−11
$\eta(3)$	−10

and

n	m	r	$\mathcal{C}_\mathbf{b}(m)\cdot\eta(\sigma^m(r))$	$\mathcal{C}_\mathbf{b}$	$S_{\mathcal{C}_\mathbf{b}}$	$H_{\mathcal{C}_\mathbf{b}}$	$D_{\mathcal{C}_\mathbf{b}}$
0	0	0	$\eta(0)$	00	0		0
1	0	1	$\eta(1)$	01	1	1	1
2	0	2	$\eta(2)$	−11	0	−1	−1
3	0	3	$\eta(3)$	−10	−1	−1	−2
4	1	0	$\mathcal{C}_\mathbf{b}(1)\cdot\eta(\sigma^1(0))$	01−10	0	1	2
5	1	1	$\mathcal{C}_\mathbf{b}(1)\cdot\eta(\sigma^1(1))$	01−11	1	1	3
6	1	2	$\mathcal{C}_\mathbf{b}(1)\cdot\eta(\sigma^1(2))$	01 01	2	1	5
7	1	3	$\mathcal{C}_\mathbf{b}(1)\cdot\eta(\sigma^1(3))$	01 00	1	−1	4
8	2	0	$\mathcal{C}_\mathbf{b}(2)\cdot\eta(\sigma^0(0))$	−11 00	0	−1	−4
9	2	1	$\mathcal{C}_\mathbf{b}(2)\cdot\eta(\sigma^0(1))$	−11 01	1	1	−3
10	2	2	$\mathcal{C}_\mathbf{b}(2)\cdot\eta(\sigma^0(2))$	−11−11	0	−1	−5
11	2	3	$\mathcal{C}_\mathbf{b}(2)\cdot\eta(\sigma^0(3))$	−11−10	−1	−1	−6
12	3	0	$\mathcal{C}_\mathbf{b}(3)\cdot\eta(\sigma^1(0))$	−10−10	−2	−1	−10
13	3	1	$\mathcal{C}_\mathbf{b}(3)\cdot\eta(\sigma^1(1))$	−10−11	−1	1	−9
14	3	2	$\mathcal{C}_\mathbf{b}(3)\cdot\eta(\sigma^1(2))$	−10 01	0	1	−7
15	3	3	$\mathcal{C}_\mathbf{b}(3)\cdot\eta(\sigma^1(3))$	−10 00	−1	−1	−8
16	4	0	$\mathcal{C}_\mathbf{b}(4)\cdot\eta(\sigma^0(0))$	01−10 00	0	1	8
17	4	1	$\mathcal{C}_\mathbf{b}(4)\cdot\eta(\sigma^0(1))$	01−10 01	1	1	9
18	4	2	$\mathcal{C}_\mathbf{b}(4)\cdot\eta(\sigma^0(2))$	01−10−11	0	−1	7
19	4	3	$\mathcal{C}_\mathbf{b}(4)\cdot\eta(\sigma^0(3))$	01−10−10	−1	−1	6
20	5	0	$\mathcal{C}_\mathbf{b}(5)\cdot\eta(\sigma^1(0))$	01−11−10	0	1	10
21	5	1	$\mathcal{C}_\mathbf{b}(5)\cdot\eta(\sigma^1(1))$	01−11−11	1	1	11
22	5	2	$\mathcal{C}_\mathbf{b}(5)\cdot\eta(\sigma^1(2))$	01−11 01	2	1	13
23	5	3	$\mathcal{C}_\mathbf{b}(5)\cdot\eta(\sigma^1(3))$	01−11 00	1	−1	12
24	6	0	$\mathcal{C}_\mathbf{b}(6)\cdot\eta(\sigma^0(0))$	01 01 00	2	1	20
25	6	1	$\mathcal{C}_\mathbf{b}(6)\cdot\eta(\sigma^0(1))$	01 01 01	3	1	21
26	6	2	$\mathcal{C}_\mathbf{b}(6)\cdot\eta(\sigma^0(2))$	01 01−11	2	−1	19
27	6	3	$\mathcal{C}_\mathbf{b}(6)\cdot\eta(\sigma^0(3))$	01 01−10	1	−1	18
28	7	0	$\mathcal{C}_\mathbf{b}(7)\cdot\eta(\sigma^1(0))$	01 00−10	0	−1	14
29	7	1	$\mathcal{C}_\mathbf{b}(7)\cdot\eta(\sigma^1(1))$	01 00−11	1	1	15
30	7	2	$\mathcal{C}_\mathbf{b}(7)\cdot\eta(\sigma^1(2))$	01 00 01	2	1	17
31	7	3	$\mathcal{C}_\mathbf{b}(7)\cdot\eta(\sigma^1(3))$	01 00 00	1	−1	16

Example 2.3. We take $\mathbf{b} = \{1, -1, -1, 1, -1, -1, \ldots\}$. Then $K = 3$,

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix},$$

$\eta(r)$	word
$\eta(0)$	0 00
$\eta(1)$	0 01
$\eta(2)$	0−11
$\eta(3)$	0−10
$\eta(4)$	−1−10
$\eta(5)$	−1−11
$\eta(6)$	−1 01
$\eta(7)$	−1 00

and

n	m	r	$\mathcal{C}_{\mathbf{b}}(m) \cdot \eta(\sigma^m(r))$	$\mathcal{C}_{\mathbf{b}}$	$S_{\mathcal{C}_{\mathbf{b}}}$	$H_{\mathcal{C}_{\mathbf{b}}}$	$D_{\mathcal{C}_{\mathbf{b}}}$
0	0	0	$\eta(0)$	0 00	0		0
1	0	1	$\eta(1)$	0 01	1	1	1
2	0	2	$\eta(2)$	0−11	0	−1	−1
3	0	3	$\eta(3)$	0−10	−1	−1	−2
4	0	4	$\eta(4)$	−1−10	−2	−1	−6
5	0	5	$\eta(5)$	−1−11	−1	1	−5
6	0	6	$\eta(6)$	−1 01	0	1	−3
7	0	7	$\eta(7)$	−1 00	−1	−1	−4
8	1	0	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(0))$	0 01−1 00	0	1	4
9	1	1	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(1))$	0 01−1 01	1	1	5
10	1	2	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(2))$	0 01−1−11	0	−1	3
11	1	3	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(3))$	0 01−1−10	−1	−1	2
12	1	4	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(4))$	0 01 0−10	0	1	6
13	1	5	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(5))$	0 01 0−11	1	1	7
14	1	6	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(6))$	0 01 0 01	2	1	9
15	1	7	$\mathcal{C}_{\mathbf{b}}(1) \cdot \eta(\sigma^1(7))$	0 01 0 00	1	−1	8
16	2	0	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(0))$	0−11 0 00	0	−1	−8
17	2	1	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(1))$	0−11 0 01	1	1	−7
18	2	2	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(2))$	0−11 0−11	0	−1	−9
19	2	3	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(3))$	0−11 0−10	−1	−1	−10
20	2	4	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(4))$	0−11−1−10	−2	−1	−14
21	2	5	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(5))$	0−11−1−11	−1	1	−13
22	2	6	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(6))$	0−11−1 01	0	1	−11
23	2	7	$\mathcal{C}_{\mathbf{b}}(2) \cdot \eta(\sigma^0(7))$	0−11−1 00	−1	−1	−12
24	3	0	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(0))$	0−10−1 00	−2	−1	−20
25	3	1	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(1))$	0−10−1 01	−1	1	−19
26	3	2	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(2))$	0−10−1−11	−2	−1	−21
27	3	3	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(3))$	0−10−1−10	−3	−1	−22
28	3	4	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(4))$	0−10 0−10	−2	1	−18
29	3	5	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(5))$	0−10 0−11	−1	1	−17
30	3	6	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(6))$	0−10 0 01	0	1	−15
31	3	7	$\mathcal{C}_{\mathbf{b}}(3) \cdot \eta(\sigma^1(7))$	0−10 0 00	−1	−1	−16

3. Proof of Theorem 1.1

Let us denote the sum of digits of the word $\eta(\sigma^m(r))$ by $\mathrm{Sum}(\eta(\sigma^m(r)))$. Then it follows from Definition 1.2 that

$$S_{\mathcal{C}_{\mathbf{b}}}(n) = S_{\mathcal{C}_{\mathbf{b}}}(m) + \mathrm{Sum}(\eta(\sigma^m(r))). \tag{3.1}$$

Lemma 3.1. *For $n \in \mathbf{N}$, put $n = 2^K m + r$, $0 \le r \le 2^K - 1$.*

(i) *If $r \ne 0$, then*

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = \mathrm{Sum}(\eta(\sigma^m(r))) - \mathrm{Sum}(\eta(\sigma^m(r-1))).$$

(ii) *If $r = 0$, then*

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = H_{\mathcal{C}_{\mathbf{b}}}\Big(\frac{n}{2^K}\Big).$$

Proof. If $r \ne 0$, then $n-1 = 2^K m + r - 1$ with $0 \le r-1 \le 2^K - 1$. Hence (3.1) and (1.3) give (i).

If $r = 0$, then $n - 1 = 2^K(m-1) + 2^K - 1$. Hence (3.1) and (1.3) give

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = S_{\mathcal{C}_{\mathbf{b}}}(m) - S_{\mathcal{C}_{\mathbf{b}}}(m-1) + \mathrm{Sum}(\eta(\sigma^m(0))) - \mathrm{Sum}(\eta(\sigma^{m-1}(2^K-1))).$$

Since $\sigma^m(0) = \sigma^{m-1}(2^K - 1)$, we obtain (ii). □

Here we use the notation $A^k \| B$, which means that A^k divides B and A^{k+1} does not divide B.

Lemma 3.2. *For an integer r with $1 \le r \le 2^K - 1$, we put $r = 2^j l$ with $2^j \| r$ and l odd. Then*

$$\mathrm{Sum}(\eta(r)) - \mathrm{Sum}(\eta(r-1)) = (-1)^{\frac{l-1}{2}} b_j.$$

Proof. Let $r = r_{K-1} \cdots r_2 r_1 r_0$ be the base-2 representations of r, where r_{K-1} may be 0. By the definition of η and Remark 1, $\eta(r)$ is expressed as

$$\eta(r) = (b_{K-1}\tau^{r_K}(r_{K-1})) \cdot (b_{K-2}\tau^{r_{K-1}}(r_{K-2})) \cdots (b_1\tau^{r_2}(r_1)) \cdot (b_0\tau^{r_1}(r_0)),$$

where $\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $r_K = 0$.

Since $2^j \| r$, the base-2 representations of r and $r-1$ are

$$r = r_{K-1} \cdots r_{j+1} 1 \underbrace{0 \cdots 0}_{j}, \quad r - 1 = r_{K-1} \cdots r_{j+1} 0 \underbrace{1 \cdots 1}_{j}.$$

Hence

$$\begin{aligned}
&\mathrm{Sum}(\eta(r)) - \mathrm{Sum}(\eta(r-1)) \\
&= b_j\tau^{r_{j+1}}(1) + b_{j-1}\tau^1(0) + b_{j-2}\tau^0(0) + \cdots + b_0\tau^0(0) \\
&- b_j\tau^{r_{j+1}}(0) - b_{j-1}\tau^0(1) - b_{j-2}\tau^1(1) - \cdots - b_0\tau^1(1) \\
&= (-1)^{r_{j+1}} b_j = (-1)^{\frac{l-1}{2}} b_j.
\end{aligned}$$

The last equality follows from the base-2 representation of $(l-1)/2$, $(l-1)/2 = r_{K-1}\cdots r_{j+1}$. □

Lemma 3.3. *For $n \in \mathbf{N}$, we put $n = (2^K)^J(2^K m + r)$ with $(2^K)^J | n$, $(2^K)^{J+1} \nmid n$ and $1 \le r \le 2^K - 1$. Moreover, for the above r, we put $r = 2^j l$ with $2^j \| r$ and l odd. Then*

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = \begin{cases} (-1)^{\frac{l-1}{2}} b_j, & \text{if } m \text{ is even}, \\ -(-1)^{\frac{l-1}{2}} b_j, & \text{if } m \text{ is odd and } j = K-1, \\ (-1)^{\frac{l-1}{2}} b_j, & \text{if } m \text{ is odd and } j \le K-2. \end{cases}$$

Proof. Lemma 3.1 gives

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = \mathrm{Sum}(\eta(\sigma^m(r))) - \mathrm{Sum}(\eta(\sigma^m(r-1))).$$

If m is even, then by Lemma 3.2,

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = \mathrm{Sum}(\eta(r)) - \mathrm{Sum}(\eta(r-1)) = (-1)^{\frac{l-1}{2}} b_j.$$

If m is odd, then

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = -(\mathrm{Sum}(\eta(2^K - r)) - \mathrm{Sum}(\eta(2^K - r - 1))).$$

Here $2^K - r = 2^j(2^{K-j} - l)$ and $2^{K-j} - l$ is odd. Hence by Lemma 3.2,

$$H_{\mathcal{C}_{\mathbf{b}}}(n) = -(-1)^{\frac{2^{K-j}-l-1}{2}} b_j = \begin{cases} -(-1)^{\frac{l-1}{2}} b_j, & \text{if } j = K-1, \\ (-1)^{\frac{l-1}{2}} b_j, & \text{if } j \le K-2. \end{cases}$$

□

Lemma 3.4. (e.g., Allouche and Shallit [1], Theorem 6.5.2) *Let $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$ be a sequence with $b_k \in \{-1, 1\}$. Let $\{P_{\mathbf{b}}(n)\}_{n=1}^{\infty}$ be the paper-folding sequence with respect to $\mathbf{b}$. For $n \in \mathbf{N}$, we put $n = 2^\alpha \beta$ with $2^\alpha \| n$ and β odd. Then*

$$P_{\mathbf{b}}(n) = (-1)^{\frac{\beta-1}{2}} b_\alpha.$$

Now we prove Theorem 1.1. Under the same notation as in Lemma 3.3, $n = (2^K)^J(2^K m + 2^j l) = 2^{KJ+j}(2^{K-j}m + l)$. Here $2^{K-j}m + l$ is odd. Hence by Lemma 3.4 and the periodicity of $\mathbf{b}$,

$$P_{\mathbf{b}}(n) = (-1)^{\frac{2^{K-j}m+l-1}{2}} b_{KJ+j} = (-1)^{\frac{2^{K-j}m+l-1}{2}} b_j,$$

which coincides with the expression of $H_{\mathcal{C}_{\mathbf{b}}}(n)$ in Lemma 3.3. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Lemma 4.1. *Denote base-2^K representation of $n \in \mathbf{N}$ by*

$$n = r_J \cdots r_2 r_1 r_0, \quad r_J \neq 0, \quad 0 \leq r_j \leq 2^K - 1, \quad j = 0, 1, 2, \ldots, J.$$

Then

$$\mathcal{C}_{\mathbf{b}}(n) = \eta(r_J) \cdot \eta(\sigma^{r_J}(r_{J-1})) \cdots \eta(\sigma^{r_2}(r_1)) \cdot \eta(\sigma^{r_1}(r_0)).$$

Proof. We prove this by induction on J.

When $J = 0$, the statement is trivial. For n with base-2^K representation $n = r_{J+1} r_J \cdots r_2 r_1 r_0$, i.e., $n = 2^K \sum_{j=0}^{J} r_{j+1}(2^K)^j + r_0$, it follows that

$$\begin{aligned}\mathcal{C}_{\mathbf{b}}(n) =& \mathcal{C}_{\mathbf{b}}\Big(\sum_{j=0}^{J} r_{j+1}(2^K)^j\Big) \cdot \eta\Big(\sigma^{\sum_{j=0}^{J} r_{j+1}(2^K)^j}(r_0)\Big) \\ =& \Big(\eta(r_{J+1}) \cdot \eta(\sigma^{r_{J+1}}(r_J)) \cdots \eta(\sigma^{r_2}(r_1))\Big) \cdot \eta(\sigma^{r_1}(r_0)),\end{aligned}$$

and this completes the proof. □

Lemma 4.2. *Let K be the minimal period of the sequence $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$. Let $\mathcal{E}$ be the set of concatenations of finitely many elements of $\{\eta(r); 0 \leq r \leq 2^K - 1\}$, where the choice of elements may be multiple and the leading elements of concatenations are different from $\eta(0)$. Then the map $\mathcal{C}_{\mathbf{b}} : \mathbf{N} \cup \{0\} \to \mathcal{E}$ is bijective.*

Proof. Lemma 4.1 and $\mathcal{C}_{\mathbf{b}}(0) = \eta(0)$ show that $\mathcal{C}_{\mathbf{b}} : \mathbf{N} \cup \{0\} \to \mathcal{E}$ is a map.

(Injectivity) Denote base-2^K representations of $n, n' \in \mathbf{N}$ by

$$n = r_J \cdots r_2 r_1 r_0, \quad r_J \neq 0, \qquad n' = r'_{J'} \cdots r'_2 r'_1 r'_0, \quad r'_{J'} \neq 0.$$

If $\mathcal{C}_{\mathbf{b}}(n) = \mathcal{C}_{\mathbf{b}}(n')$, then by Lemma 4.1, we have $J = J'$ and

$$\begin{aligned}&\eta(r_J) \cdot \eta(\sigma^{r_J}(r_{J-1})) \cdots \eta(\sigma^{r_2}(r_1)) \cdot \eta(\sigma^{r_1}(r_0)) \\ &= \eta(r'_J) \cdot \eta(\sigma^{r'_J}(r'_{J-1})) \cdots \eta(\sigma^{r'_2}(r'_1)) \cdot \eta(\sigma^{r'_1}(r'_0)).\end{aligned}$$

By the injectivity of η, this is equivalent to the system of equations

$$\begin{cases} r_J = r'_J, \\ \sigma^{r_J}(r_{J-1}) = \sigma^{r'_J}(r'_{J-1}), \\ \quad\vdots \\ \sigma^{r_1}(r_0) = \sigma^{r'_1}(r'_0), \end{cases}$$

and this yields $n = n'$.

(Surjectivity) Let $E \in \mathcal{E}$ and $E \neq \eta(0)$. Then there exist a $J \in \mathbf{N}$ and integers t_j, $j = 0, 1, 2, \ldots, J$, $t_J \neq 0$, $0 \leq t_j \leq 2^K - 1$, such that $E = \eta(t_J) \cdot \eta(t_{J-1}) \cdots \eta(t_1) \cdot \eta(t_0)$. We can find an $n \in \mathbf{N}$ with base-2^K representation $n = r_J r_{J-1} \cdots r_1 r_0$ such that

$$\eta(t_J) \cdot \eta(t_{J-1}) \cdots \eta(t_1) \cdot \eta(t_0) = \eta(r_J) \cdot \eta(\sigma^{r_J}(r_{J-1})) \cdots \eta(\sigma^{r_2}(r_1)) \cdot \eta(\sigma^{r_1}(r_0)).$$

In fact, by the injectivity of η, this is equivalent to the system of equations

$$\begin{cases} t_J = r_J, \\ t_{J-1} = \sigma^{r_J}(r_{J-1}), \\ \qquad \vdots \\ t_0 = \sigma^{r_1}(r_0), \end{cases}$$

and hence $r_J, \dots, r_0$ are uniquely determined. Thus $E = \mathcal{C}_{\mathbf{b}}(n)$. □

For $n \in \mathbf{N} \cup \{0\}$, let us express the nth word of $\mathcal{C}_{\mathbf{b}}$ as $\mathcal{C}_{\mathbf{b}}(n) = u_J \cdots u_2 u_1 u_0$. Then $D_{\mathcal{C}_{\mathbf{b}}} : \mathbf{N} \cup \{0\} \to \mathbf{Z}$ is the function such that

$$D_{\mathcal{C}_{\mathbf{b}}}(n) = \sum_{j=0}^{J} u_j 2^j,$$

which was already defined in Introduction.

Lemma 4.3. *The function $D_{\mathcal{C}_{\mathbf{b}}}$ is injective.*

Proof. Assume that there exist $n, n' \in \mathbf{N} \cup \{0\}$, $n \neq n'$, such that $D_{\mathcal{C}_{\mathbf{b}}}(n) = D_{\mathcal{C}_{\mathbf{b}}}(n')$. Denote $\mathcal{C}_{\mathbf{b}}(n) = u_J \cdots u_2 u_1 u_0$ and $\mathcal{C}_{\mathbf{b}}(n') = u'_{J'} \cdots u'_2 u'_1 u'_0$. We may suppose that these words have the same length by supplying enough 0's on the left of words. Then there exists an $L \in \mathbf{N} \cup \{0\}$ such that $u_L \neq u'_L$ and $u_j = u'_j$ for all $j \geq L+1$. Then $D_{\mathcal{C}_{\mathbf{b}}}(n) = D_{\mathcal{C}_{\mathbf{b}}}(n')$ gives

$$(u_L - u'_L) 2^L = \sum_{j=0}^{L-1} (u'_j - u_j) 2^j,$$

and hence

$$|u_L - u'_L| 2^L \leq \sum_{j=0}^{L-1} |u'_j - u_j| 2^j.$$

If $u'_j = 0$, then $u_j = 1$ or -1, and if $u'_j = 1$ or -1, then $u_j = 0$ (cf. Examples 2.1, 2.2, and 2.3). Hence $|u'_j - u_j| = 1$, and this gives $2^L \leq 2^L - 1$, which is a contradiction. □

For $0 \leq n \leq 2^K - 1$, let us express the nth word of RBC as $\mathrm{RBC}(n) = s_{K-1} \cdots s_2 s_1 s_0$, where s_{K-1} may be 0. Then by Definitions 1.1 and 1.2,

$$\mathcal{C}_{\mathbf{b}}(n) = (b_{K-1} s_{K-1}) \cdots (b_2 s_2) \cdot (b_1 s_1) \cdot (b_0 s_0).$$

Thus, for $0 \leq n \leq 2^K - 1$,

$$D_{\mathcal{C}_{\mathbf{b}}}(n) = \sum_{j=0}^{K-1} b_j s_j 2^j. \tag{4.1}$$

Now let us define the set $\mathcal{D}$ by

$$\mathcal{D} = \mathcal{D}_{\mathbf{b}} = \{ D_{\mathcal{C}_{\mathbf{b}}}(n) : 0 \leq n \leq 2^K - 1 \}.$$

Lemma 4.4. *The set $\mathcal{D}$ has the following properties*:

(i) $\mathcal{D} \subset \mathbf{Z}$.
(ii) $0, 1 \in \mathcal{D}$.
(iii) $\mathcal{D}$ *contains* 2^K *elements.*
(iv) *If* $d \in \mathcal{D}$, *then* $-2^K + 1 \leq d \leq 2^K - 1$.
(v) *If* $d, d' \in \mathcal{D}$ *and* $d \neq d'$, *then* $d \not\equiv d' \pmod{2^K}$.
(vi) $\max_{d,d' \in \mathcal{D}} |d - d'| = 2^K - 1$, *i.e., the set* $\mathcal{D}$ *has no gap.*
(vii) *If* $K \geq 2$, *then* $-1 \in \mathcal{D}$.

Proof. (i) – (iv) are obvious.

(v) Assume that $D_{\mathcal{C}_{\mathbf{b}}}(n) \equiv D_{\mathcal{C}_{\mathbf{b}}}(n') \pmod{2^K}$. By the same notation and argument as in the proof of Lemma 4.3, we have

$$(u_L - u'_L)2^L - M2^K = \sum_{j=0}^{L-1} (u'_j - u_j)2^j, \quad M \in \mathbf{Z}$$

with the additional condition $0 \leq L \leq K - 1$. Taking the absolute values on both sides, we have

$$2^L \leq ||M|2^K - 2^L| \leq \sum_{j=0}^{L-1} 2^j = 2^L - 1,$$

which is a contradiction.

(vi) When $\mathbf{b} = \{1, 1, 1, 1, \ldots\}$, i.e., $K = 1$, then $\mathcal{D}_{\mathbf{b}} = \{0, 1\}$ and $\max_{d,d' \in \mathcal{D}_{\mathbf{b}}} |d - d'| = 1$. Assume that the statement is true for all $\mathbf{b} = \{b_0, b_1, \ldots, b_{K-1}, \cdots\}$. Let l and m be the minimum and the maximum of $\mathcal{D}_{\mathbf{b}}$, respectively. Then $m - l = 2^K - 1$.

Let us consider $\widetilde{\mathbf{b}} = \{b_0, b_1, \ldots, b_{K-1}, \beta_K, \cdots\}$ with $\beta_K = 1$ or -1. From (4.1) it follows that $D_{\mathcal{C}_{\widetilde{\mathbf{b}}}} = \beta_K s_K 2^K + D_{\mathcal{C}_{\mathbf{b}}}$, and hence $\mathcal{D}_{\widetilde{\mathbf{b}}} = \mathcal{D}_{\mathbf{b}} \cup (\beta_K 2^K + \mathcal{D}_{\mathbf{b}})$.

If $\beta_K = 1$, then the maximum of $\mathcal{D}_{\mathbf{b}}$ is $l + 2^K - 1$, and the minimum of $\beta_K 2^K + \mathcal{D}_{\mathbf{b}}$ is $2^K + l$. Hence $\max_{d,d' \in \mathcal{D}_{\widetilde{\mathbf{b}}}} |d - d'| = 2^{K+1} - 1$.

If $\beta_K = -1$, then the minimum of $\mathcal{D}_{\mathbf{b}}$ is $m - 2^K + 1$, and the maximum of $\beta_K 2^K + \mathcal{D}_{\mathbf{b}}$ is $-2^K + m$. Hence $\max_{d,d' \in \mathcal{D}_{\widetilde{\mathbf{b}}}} |d - d'| = 2^{K+1} - 1$.

(vii) By (i) (iii) (iv) (v), either $2^K - 1$ or -1 belongs to $\mathcal{D}$. Assume that $2^K - 1$ belongs to $\mathcal{D}$. Then, by (4.1), there exists an $n \in \mathbf{N} \cup \{0\}$ with $0 \leq n \leq 2^K - 1$ and $\mathcal{C}_{\mathbf{b}}(n) = (b_{K-1}s_{K-1}) \cdots (b_2 s_2) \cdot (b_1 s_1) \cdot (b_0 s_0)$ such that

$$\sum_{j=0}^{K-1} 2^j = \sum_{j=0}^{K-1} b_j s_j 2^j.$$

Since $b_j = 1$ or -1, and $s_j = 1$ or 0, the above equality gives $b_j = 1$ for all $0 \leq j \leq K - 1$. Hence $K = 1$, which is a contradiction. $\square$

Lemma 4.5. *Let $K \geq 2$ and $z_0 \in \mathbf{Z}$. Let us consider the algorithm*

$$\begin{aligned} z_0 &\equiv d_0 \pmod{2^K}, & z_0 &= d_0 + 2^K z_1, \\ z_1 &\equiv d_1 \pmod{2^K}, & z_1 &= d_1 + 2^K z_2, \\ z_2 &\equiv d_2 \pmod{2^K}, & z_2 &= d_2 + 2^K z_3, \\ &\vdots & &\vdots \end{aligned}$$

where d_0, d_1, d_2, ... are chosen as elements of $\mathcal{D}$. Then the following properties hold:

(i) $z_1, z_2, z_3, \ldots$ *are uniquely determined.*

(ii) *If $z_0 \geq 0$, then there exists an $L \in \mathbf{N} \cup \{0\}$ such that $z_0 > z_1 > z_2 > \cdots > z_L = 0$.*

(iii) *If $z_0 \leq 0$, then there exists an $L \in \mathbf{N} \cup \{0\}$ such that $z_0 < z_1 < z_2 < \cdots < z_L = 0$.*

Hence $z_0 = \sum_{l=0}^{L-1} d_l (2^K)^l$ with $d_{L-1} \neq 0$, and this is identified with the representation $z_0 = d_{L-1} \cdots d_2 \cdot d_1 \cdot d_0$.

Proof. (i) By Lemma 4.4 (i) (iii) (v), $\mathcal{D}$ is the set of all residue classes modulo 2^K, and hence $z_1, z_2, z_3, \ldots$ are uniquely determined.

(ii) If $z_0 \geq 0$ belongs to $\mathcal{D}$, then d_0 is z_0. Hence $z_1 = 0$.

If $z_0 \geq 0$ does not belong to $\mathcal{D}$, then z_0 is greater than all elements of $\mathcal{D}$ by Lemma 4.4 (ii) (vi). Hence $z_1 = (z_0 - d_0)/2^K > 0$. Moreover, from Lemma 4.4 (iv) and $z_0 \geq 2$, it follows that

$$z_1 \leq \frac{z_0 + 2^K - 1}{2^K} < \frac{z_0}{2} + 1 \leq z_0.$$

Hence $z_0 > z_1 > 0$.

Repeating the above argument, we have the result.

(iii) This is similarly proved as (ii). □

To complete the proof of Theorem 1.2, we prove that $D_{\mathcal{C}_\mathbf{b}}$ is surjective. By Lemma 4.5 any $z_0 \in \mathbf{Z}$ has base-2^K representation $z_0 = d_{L-1} \cdots d_2 \cdot d_1 \cdot d_0$ with $d_j \in \mathcal{D}$, $0 \leq j \leq L-1$, and $d_{L-1} \neq 0$. By (4.1) and the definition of η, each d_j has the base-2 representation $d_j = \eta(t_j)$, $0 \leq t_j \leq 2^K - 1$. Hence z_0 has the representation

$$z_0 = \eta(t_{L-1}) \cdots \eta(t_2) \cdot \eta(t_1) \cdot \eta(t_0), \quad t_{L-1} \neq 0.$$

By this representation and Lemma 4.2, there exists an $n \in \mathbf{N} \cup \{0\}$ such that $z_0 = \sum_{j=0}^{J} u_j 2^j$, where u_j are the digits of $\mathcal{C}_\mathbf{b}(n) = u_J \cdots u_2 u_1 u_0$. Thus, $D_{\mathcal{C}_\mathbf{b}}$ is surjective.

This completes the proof of Theorem 1.2.

Here we give two examples.

Example 4.1. We take $\mathbf{b} = \{1, -1, 1, -1, \ldots\}$. Then $K = 2$,

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix},$$

r	$\eta(r)$	$D_{\mathcal{C}_\mathbf{b}}(r)$
0	$\eta(0) = 00$	0
1	$\eta(1) = 01$	1
2	$\eta(2) = -11$	-1
3	$\eta(3) = -10$	-2

and $\mathcal{D} = \{-2, -1, 0, 1\}$. We take a positive value 14 and calculate the n such that $D_{\mathcal{C}_\mathbf{b}}(n) = 14$:

$$\begin{aligned} 14 &\equiv -2 \pmod{2^2}, \quad 14 = -2 + 2^2 \times 4, \\ 4 &\equiv 0 \pmod{2^2}, \quad 4 = 0 + 2^2 \times 1, \\ 1 &\equiv 1 \pmod{2^2}, \quad 1 = 1 + 2^2 \times 0. \end{aligned}$$

Hence

$$\begin{aligned} 14 =& (1) \cdot (0) \cdot (-2) = (01) \cdot (00) \cdot (-10) \\ =& \eta(1) \cdot \eta(0) \cdot \eta(3) = \eta(r_2) \cdot \eta(\sigma^{r_2}(r_1)) \cdot \eta(\sigma^{r_1}(r_0)) \end{aligned}$$

for an n with the base-2^2 representation $n = r_2 r_1 r_0$. Solving the system of equations

$$\begin{cases} 1 = r_2, \\ 0 = \sigma^{r_2}(r_1), \\ 3 = \sigma^{r_1}(r_0), \end{cases}$$

we have $r_2 r_1 r_0 = 130$, i.e., $n = 28$. Hence $\mathcal{C}_\mathbf{b}(28) = 0100{-}10$ and $D_{\mathcal{C}_\mathbf{b}}(28) = 14$.

Example 4.2. We take $\mathbf{b} = \{1, -1, -1, 1, -1, -1, \ldots\}$. Then $K = 3$,

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{pmatrix},$$

r	$\eta(r)$	$D_{\mathcal{C}_\mathbf{b}}(r)$
0	$\eta(0) = 0\ 00$	0
1	$\eta(1) = 0\ 01$	1
2	$\eta(2) = 0{-}11$	-1
3	$\eta(3) = 0{-}10$	-2
4	$\eta(4) = -1{-}10$	-6
5	$\eta(5) = -1{-}11$	-5
6	$\eta(6) = -1\ 01$	-3
7	$\eta(7) = -1\ 00$	-4

and $\mathcal{D} = \{-6, -5, -4, -3, -2, -1, 0, 1\}$. We take a negative value -21 and calculate the n such that $D_{\mathcal{C}_\mathbf{b}}(n) = -21$:

$$\begin{aligned} -21 &\equiv -5 \pmod{2^3}, \quad -21 = -5 + 2^3 \times (-2), \\ -2 &\equiv -2 \pmod{2^3}, \quad -2 = -2 + 2^3 \times 0. \end{aligned}$$

Hence

$$\begin{aligned}-21 =&(-2)\cdot(-5) = (0{-}10)\cdot(-1{-}11)\\ =&\eta(3)\cdot\eta(5) = \eta(r_1)\cdot\eta(\sigma^{r_1}(r_0))\end{aligned}$$

for an n with the base-2^3 representation $n = r_1r_0$. Solving the system of equations

$$\begin{cases}3 = r_1,\\ 5 = \sigma^{r_1}(r_0),\end{cases}$$

we have $r_1r_0 = 32$, i.e., $n = 26$. Hence $\mathcal{C}_{\mathbf{b}}(26) = 0{-}10{-}1{-}11$ and $D_{\mathcal{C}_{\mathbf{b}}}(26) = -21$.

5. Proof of Theorem 1.3

Firstly, we prepare some lemmas.

Lemma 5.1. *Let K be the minimal period of the sequence $\mathbf{b} = \{b_k\}_{k=0}^{\infty}$. Let $\mathcal{C}_{\mathbf{b}}$ be the code induced by $\mathbf{b}$, and $S_{\mathcal{C}_{\mathbf{b}}}$ be the sum of digits function for $\mathcal{C}_{\mathbf{b}}$. Let $f = \Psi_{2^K}(S_{\mathcal{C}_{\mathbf{b}}})$ (cf. (1.4)). Then the following properties hold:*

(i) [Periodicity]: $f(n + 2^{K+1}) = f(n)$.

(ii) $f(2^{K+1} - n) = -f(n), \quad 1 \le n \le 2^K - 1$.

(iii) [Zero-sum]: $\sum_{n=0}^{2^{K+1}-1} f(n) = 0$.

(iv) *It follows that*

$$f(n) = \sum_{k=0}^{K-1} b_k f_0\Big(\frac{n}{2^k}\Big), \tag{5.1}$$

where $f_0(x) : [0,\infty) \to \mathbf{Z}$ is defined as follows: if $x = n \in \mathbf{Z}$, then

$$f_0(n) = \begin{cases}0, & \text{if } n \equiv 0 \pmod 4,\\ 1, & \text{if } n \equiv 1 \pmod 4,\\ 0, & \text{if } n \equiv 2 \pmod 4,\\ -1, & \text{if } n \equiv 3 \pmod 4,\end{cases}$$

which is a Dirichlet character modulo 4, and, if $x \notin \mathbf{Z}$, then $f_0(x) = 0$.

Proof. (i) If $n \not\equiv 0 \pmod{2^K}$, then by Lemma 3.1 (i),

$$f(n) = \mathrm{Sum}(\eta(\sigma^m(r))) - \mathrm{Sum}(\eta(\sigma^m(r-1))),$$

where $n = 2^K m + r$. Then $n + 2^{K+1} = 2^K(m+2) + r$, Lemma 3.1 (i), and $\sigma^2 = \mathrm{id}$ give $f(n + 2^{K+1}) = f(n)$.

If $n \equiv 0 \pmod{2^K}$, then Lemma 3.1 (ii) gives $f(n + 2^{K+1}) = f(n) = 0$.
(ii) For $1 \le n \le 2^K - 1$, Lemma 3.1 (i) gives that

$$f(n) = \mathrm{Sum}(\eta(n)) - \mathrm{Sum}(\eta(n-1))$$

and

$$f(2^{K+1}-n)=\mathrm{Sum}(\eta(\sigma(2^K-n)))-\mathrm{Sum}(\eta(\sigma(2^K-n-1))).$$

Since $\sigma(2^K-n)=n-1$, we have

$$f(2^{K+1}-n)=\mathrm{Sum}(\eta(n-1))-\mathrm{Sum}(\eta(n))=-f(n).$$

(iii) This is a consequence of (ii).
(iv) When $n=0,2^K$, both sides of (5.1) are 0. For $1\le n\le 2^K-1$, Lemma 3.1 (i) and Lemma 3.2 give

$$f(n)=\mathrm{Sum}(\eta(n))-\mathrm{Sum}(\eta(n-1))=(-1)^{\frac{l-1}{2}}b_j,$$

where $2^j\|n$, $n=2^jl$. On the other hand, for $1\le n\le 2^K-1$, by the definition of f_0,

$$\sum_{k=0}^{K-1}b_kf_0\Big(\frac{n}{2^k}\Big)=b_jf_0(l)=(-1)^{\frac{l-1}{2}}b_j.$$

Hence (5.1) holds for $0\le n\le 2^K$. For $2^K+1\le n\le 2^{K+1}-1$, we have by (ii) that

$$f(n)=-f(2^{K+1}-n)=-\sum_{k=0}^{K-1}b_kf_0\Big(\frac{2^{K+1}-n}{2^k}\Big),$$

and it is easy to verify that $f_0(\frac{2^{K+1}-n}{2^k})=-f_0(\frac{n}{2^k})$. Hence (5.1) holds for $0\le n\le 2^{K+1}-1$, and this completes the proof. □

The following is a consequence of Lemma 5.1.

Lemma 5.2. *Under the same notation as in Lemma* 5.1*, let* $\xi:[0,\infty)\to\mathbf{C}$ *be the function such that* $\xi(x)=\sum_{0\le n\le x}f(n)$*. Then the following properties hold*:

(i) $\xi(x+2^{K+1})=\xi(x)$.
(ii) $\xi(2^{K+1}-x)=\xi(x),\quad 0\le x\le 2^{K+1},\ x\notin\mathbf{Z}$.
(iii) *It follows that*

$$\xi(x)=\sum_{k=0}^{K-1}b_k\xi_0\Big(\frac{x}{2^k}\Big),$$

where $\xi_0:[0,\infty)\to\mathbf{Z}$ *is defined as follows*: *for* $0\le x<4$,

$$\xi_0(x)=\begin{cases}0, & \text{if } 0\le x<1,\\ 1, & \text{if } 1\le x<3,\\ 0, & \text{if } 3\le x<4,\end{cases}$$

and, for $x\ge 4$, $\xi_0(x)$ *is defined to be periodic with period* 4.

Let $g\in\mathcal{A}$, and $p\ge 2$ be an integer. We assume two properties on g:

[Periodicity]: g is a periodic function with period p,
[Zero-sum]: $\sum_{n=0}^{p-1} g(n) = 0$.

From [Periodicity] and [Zero-sum] the Dirichlet series

$$L(s,g) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \quad \Re s > 1$$

can be analytically extended to the whole complex s-plane, and this extension is also denoted by $L(s,g)$. For the average of $S = \Phi_q(g)$, we can prove the following by the same argument as in Section 7 of [3].

Lemma 5.3. *Let $p, q \geq 2$ be integers with $q|p$. Assume that $g \in \mathcal{A}$ satisfies* [Periodicity] *and* [Zero-sum]. *Let $S = \Phi_q(g)$. Let $\xi : [0,\infty) \to \mathbf{C}$ be the function such that $\xi(x) = \sum_{0 \leq n \leq x} g(n)$. Then*

$$L(0,g) = \frac{1}{p}\int_0^p \xi(x)dx, \tag{5.2}$$

and for any positive integer N,

$$\frac{1}{N}\sum_{n=0}^{N-1} S(n) = \frac{\log N}{\log q} L(0,g) + F\Big(\frac{\log N}{\log q}\Big) - \frac{1}{N} G(N)$$

with the following F and G.

The function $F : [0,\infty) \to \mathbf{C}$ is defined by either of the following two ways (I) *and* (II), *and $F(x)$ is periodic with period* 1 *and continuous*:

(I) $\quad F(x) = (1 + \lfloor x \rfloor - x)L(0,g) + q^{\lfloor x \rfloor - x} \sum_{r=0}^{\infty} \int_0^{q^{x - \lfloor x \rfloor}} (\xi(q^r t) - L(0,g))dt,$

(II) $\quad F(x) = \sum_{k \in \mathbf{Z}} D_k e^{2\pi i k x}$ *whose Fourier coefficients are given by*

$$\begin{cases} D_0 = \Big(\frac{1}{2} - \frac{1}{\log q}\Big)L(0,g) + \frac{L'(0,g)}{\log q}, & \\ D_k = \frac{L(\frac{2\pi i k}{\log q}, g)}{2\pi i k(\frac{2\pi i k}{\log q} + 1)}, & k \neq 0. \end{cases}$$

The function $G : \mathbf{N} \to \mathbf{C}$ is defined by

$$G(N) = \sum_{r=1}^{\infty} \frac{1}{q^r} \int_0^{q^r N} (\xi(x) - L(0,g))dx,$$

and $G(N)$ is periodic with period p/q, i.e., $G(N + p/q) = G(N)$.

Now we prove Theorem 1.3. By Lemma 5.1, $f = \Psi_{2^K}(S_{\mathcal{C}_{\mathbf{b}}})$ satisfies the assumptions [Periodicity] and [Zero-sum] with $p = 2^{K+1}$ and $q = 2^K$. Hence we have the expression of $\frac{1}{N}\sum_{n=0}^{N-1} S_{\mathcal{C}_{\mathbf{b}}}(n)$ (cf. Theorem 1.3) corresponding to that of Lemma 5.3. We need to calculate $L(0,f)$ and prove $G(N) = 0$.

Lemma 5.4. *Under the same notation as in Theorem* 1.3*,*

$$L(0,f)=\frac{1}{2}\sum_{k=0}^{K-1}b_k,$$

and $G(N)=0$ *for any* $N\in\mathbf{N}$.

Proof. By (5.2) with $f=\Psi_{2^K}(S_{\mathcal{C}_{\mathbf{b}}})$ and $p=2^{K+1}$, Lemma 5.2 (iii), and the definition of ξ_0,

$$\begin{aligned}L(0,f)=&\frac{1}{2^{K+1}}\sum_{k=0}^{K-1}b_k2^k\int_0^{2^{K+1-k}}\xi_0(x)dx\\=&\frac{1}{2^{K+1}}\sum_{k=0}^{K-1}b_k2^k\left(\int_0^{2^{K+1-k}}\Big(\xi_0(x)-\frac{1}{2}\Big)dx+\frac{1}{2}\int_0^{2^{K+1-k}}dx\right)\\=&\frac{1}{2^{K+1}}\sum_{k=0}^{K-1}b_k2^k2^{K-k}=\frac{1}{2}\sum_{k=0}^{K-1}b_k.\end{aligned}$$

This gives

$$\int_0^{2^{K+1}}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx=0. \tag{5.3}$$

By (5.3) and Lemma 5.2 (ii),

$$\int_0^{2^{K}}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx=0. \tag{5.4}$$

If $M=2M'$, $M'\in\mathbf{N}$, then by (5.3) and Lemma 5.2 (i),

$$\int_0^{2^{K}(2M')}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx=0. \tag{5.5}$$

If $M=2M'+1$, then by (5.5), Lemma 5.2 (i), and (5.4),

$$\begin{aligned}\int_0^{2^K(2M'+1)}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx&=\int_{2^{K+1}M'}^{2^{K+1}M'+2^K}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx\\&=\int_0^{2^K}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx=0.\end{aligned}$$

Hence for any $M\in\mathbf{N}$,

$$\int_0^{2^KM}\Big(\xi(x)-\frac{1}{2}\sum_{k=0}^{K-1}b_k\Big)dx=0,$$

and this gives $G(N)=0$. □

We now have

$$\sum_{n=0}^{N-1} S_{\mathcal{C}_{\mathfrak{b}}}(n) = \frac{N\log N}{\log 2^K} \times B + N \times F\Big(\frac{\log N}{\log 2^K}\Big), \tag{5.6}$$

where $B = \frac{1}{2}\sum_{k=0}^{K-1} b_k$ and $F(x)$ is the periodic function with period 1. We can prove $L(\frac{2\pi ik}{\log 2^K}, f) \ll (1+|k|)^{1/2}$ by the same argument as in the proof of Lemma 7.4 of [3]. Hence the Fourier series in Theorem 1.3 is absolutely and uniformly convergent, and this yields that $F(x)$ is continuous.

To complete the proof of Theorem 1.3, we prove that $F(x)$ is nowhere differentiable. The proof is similar to that of Tenenbaum [4] (see also Allouche and Shallit [1], Theorem 3.5.3 (ii)).

Let $\lambda, r \in \mathbf{N}$, and $x \in (0,1)$. Let $0 \le \varepsilon_h \le 2^K - 1$ be the integer such that

$$(2^K)^x = \sum_{h=0}^{\infty} \varepsilon_h (2^K)^{-h},$$

and N_i be the positive integer such that

$$N_i = N_i(\lambda, r) = (2^K)^{\lambda} \sum_{h=0}^{i} \varepsilon_h (2^K)^{i-h} + r.$$

Let

$$x_i = \frac{\log N_i}{\log 2^K} - i - \lambda, \quad y_i = \frac{\log(N_i+1)}{\log 2^K} - i - \lambda.$$

These give

$$\frac{\log N_i}{\log 2^K} - i - \lambda = x + o(1), \quad i \to \infty,$$

$$x_i, y_i \to x, \quad i \to \infty,$$

and

$$y_i - x_i = \frac{1}{N_i \log 2^K}\Big(1 + O\Big(\frac{1}{N_i}\Big)\Big).$$

Assume that $F(x)$ is differentiable. Then by (5.6) and the same argument as in [4], we have

$$S_{\mathcal{C}_{\mathfrak{b}}}(N_i) = \frac{((x+i+\lambda)\log 2^K + 1)B}{\log 2^K} + \frac{F'(x)}{\log 2^K} + F(x) + o(1), \quad i \to \infty.$$

Applying the above asymptotic formula to the cases that $N_i = N_i(1,r)$, $N_i(1, r-1)$ with $1 \le r \le 2^K - 1$, we get

$$H_{\mathcal{C}_{\mathfrak{b}}}(N_i(1,r)) = o(1), \quad i \to \infty. \tag{5.7}$$

On the other hand, by Lemma 3.3, $H_{\mathcal{C}_{\mathfrak{b}}}(N_i(1,r))$ takes only the values 1 and -1, and hence (5.7) does not hold. This is a contradiction.

References

[1] J.-P. Allouche and J. Shallit, *Automatic Sequences*, Theory, Applications, Generalizations, Cambridge University Press, Cambridge, (2003).
[2] H. Delange, *Sur la fonction sommatoire de la fonction "somme des chiffres"*, L'Enseignement Math., **21**, (1975), 31–47.
[3] Y. Kamiya and L. Murata, *Relations among arithmetical functions, automatic sequences, and sum of digits functions induced by certain Gray codes*, J. Théor. Nombres Bordx., **24**, 2 (2012), 307–337.
[4] G. Tenenbaum, *Sur la non-dérivabilité de fonctions périodiques associées à certaines formules sommatoires*, in The Mathematics of Paul Erdős, R. L. Graham and J. Nešetřil eds., Springer Verlag, (1997), 117–128.

Yuichi Kamiya
Department of Modern Economics
Faculty of Economics
Daito Bunka University
560 Iwadono, Higashi-Matsuyama
Saitama 355-8501, Japan
E-mail: ykamiya@ic.daito.ac.jp

Leo Murata
Department of Mathematics
Faculty of Economics
Meiji Gakuin University
1-2-37 Shirokanedai
Minato-ku, Tokyo 108-8636, Japan
E-mail: leo@eco.meijigakuin.ac.jp