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Résumé. *Axel Thue en contexte.*
Les travaux d'Axel Thue, en particulier son célèbre article "Über Annäherungswerte algebraischer Zahlen", sont d'ordinaire décrits comme des joyaux isolés dans les mathématiques de leur temps. Je montre ici qu'il est nécessaire de distinguer entre l'isolement personnel de Thue et les caractéristiques de sa pratique mathématique. Après une brève présentation de la biographie d'Axel Thue, je contextualise certains aspects de son travail (en particulier la conversion d'expressions sur des nombres en expressions sur des polynômes) à la lumière de son éducation mathématique spécifique et de l'état de l'art en approximation diophantienne au tournant du vingtième siècle.

Abstract. Axel Thue's works, in particular his celebrated paper, "Über Annäherungswerte algebraischer Zahlen," are usually perceived as solitary gems in the mathematics of their time. I argue here that it is important to distinguish between his personal isolation and the characteristics of his mathematical practice. While sketching out Axel Thue's biography, I shall contextualize some features of his work (in particular the conversion from expressions on numbers to expressions on polynomials) with respect to his mathematical education, as well as to the state-of-the-art Diophantine analysis and rational approximation at the turn of the twentieth century.

When the first (and until now only) volume of his *Selected Papers* was published in 1977, 55 years after his death, "Thue's name [was] known mostly for his theorem on diophantine approximation and on diophantine equations," written "when he was well in his forties and when he had been away from the centers of mathematics for over a decade," [38, p. 919]. Thue's predilection for publishing mostly in Norwegian journals may have

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Mots clefs. Axel Thue, history of mathematics, science in Norway, Diophantine approximation.
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been a factor in the restricted knowledge of his work among his contemporaries. Among the 35 bibliographical entries reproduced in his *Selected Papers*, 24 were written in German, but published in a Danish or a Norwegian journal, mostly in *Arkiv for Mathematik og Naturvidenskab* or in the Memoirs of the Kristiana Society of Sciences, *Skrifter utgit av Videnskapsselskapet i Kristiania*; 9 were written in Norwegian,[1] also for Norwegian journals, including three entitled "Mindre meddelelser (Minor communications)," grouping together resp. 3, 6 and 8 short contributions. Only 2 of his articles were published outside Scandinavia, in German, in the famous Berlin-based *Journal für die reine und angewandte Mathematik*: Thue's celebrated 1909 paper on the approximation of algebraic numbers, [55], which will be described later in detail, and a related work one year later.[2]

But all his biographers insist on Thue's isolation and, as a cause or as a consequence, on his originality and mathematical independence. "Thue was a man who would begin at the beginning and construct everything until the end," writes Karl Egil Aubert in a paper on Diophantine equations in Norway [2, p. 155]. Exactly in the same direction, one might mention Vilhelm Bjerknes's description[3]: "[Thue] never progressed very far with the reading of a mathematical work before he jettisoned it in order to rebuild its whole intellectual construction according to his own ideas." In particular, while Sophus Lie, probably the most famous Norwegian mathematician in Thue's youth, was, in part, his mentor, the lack of influence on Thue of Lie's program, and more generally of Lie himself is always underscored. Again, in Aubert's words, "Lie would not affect Thue in any appreciable degree. [Thue] was not a man to be a subordinate [...] and to provide only a small and rather ordinary addition to another man's work."

With regard to number theory, this image has been reinforced by Carl Ludwig Siegel's magisterial reappropriation of Thue's results, [46] and [47, 48]. As Siegel explained, [48, pp. xxx-xxxi], he became aware of Thue's proof that the equation $x^n - dy^n = 1$, for a fixed exponent $n > 2$, and d an integer, has only finitely many integral solutions, through a remark of Issai Schur after a talk during Siegel's third semester at Berlin university (winter semester 1916–17). "When I tried then to read [Thue's article]," Siegel went on, "I soon ended in confusion because of the numerous letters c, k, θ, ω, m, n, a, s, the deeper meaning of which seemed enigmatic to me. In order to be able to understand a bit more, though, I changed the ordering of the lemmas, introduced new symbols too, and among them,

[1] Or, more precisely, in the language close to Danish used then by the Norwegian urban elite, officialized as Riksmål, then Bokmål after Thue's death.

[2] A longer list of Thue's articles, given in [36, pp. 46-49], includes 11 more articles in Norwegian and 2 more in German, all published in Scandinavian journals, see Figure 0.1.

[3] It is reproduced in English in the introduction to Thue's *Selected Mathematical Papers*, [57, p. xxiii].

more by chance than by any deliberate thought, was a parameter which does not occur in Thue's paper, and which, to my amazement, provided a sharpening of the approximation theorem."[4] Siegel thus offered a new and larger vista on Thue's work, which certainly contributed in bringing it into the mainstream, but also blurred some recurrent characteristics of Thue's approach. The main objective of the present paper is to reconstitute some of these and to inscribe them in the context of other pieces of mathematics which informed Thue's views on number theory and rational approximation.

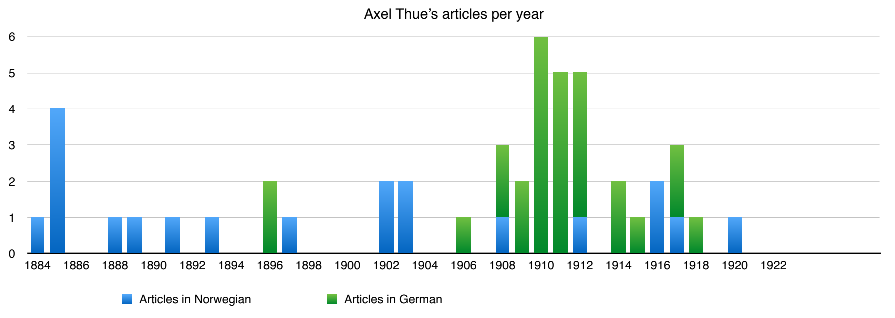


FIGURE 0.1.

1. Axel Thue and Elling Holst

Axel Thue's biography is quite straightforward.[5] In a nutshell: born on February 19, 1863, he studied mathematics and natural sciences in Norway (mainly in Kristiania).[6] At the end of the 1880s, he went abroad for the then-usual study trip to one of the most prominent scientific countries of the time, namely Germany. In 1892 he obtained a position as Assistant Professor at the Royal Frederick University in Kristiania (*Det Kongelige Frederiks Universitet*, the former name of the present day University of

[4][48, pp. xxx-xxxi]: "Als ich dann diese zu lesen versuchte, kam ich bald in Verwirrung durch die vielen Buchstaben c, k, θ, ω, m, n, a, s, deren tiefere Bedeutung mir rätselhaft schien. Um nun doch etwas mehr verstehen zu können, änderte ich die Anordnung der Hilfssätze, führte auch neue Symbole ein, und unter diesen war dann, weniger durch geordnetes Denken als durch Zufall, noch ein bei Thue nicht aufgetretener Parameter, der zu meiner Verwunderung eine Verschärfung des Approximationsatzes ergab." Despite Schur's lack of interest in this result and the Great War, which sent Siegel, a conscientious objector, to a psychiatric institute, this work would become Siegel's 1920 doctoral thesis under Edmund Landau, [46]; it would also play an important role in Siegel's contact with André Weil, [59].

[5]I am essentially reproducing here the first paragraph of the introduction of [57] by Viggo Brun.

[6]Kristiania (spelled Christiania, after King Christian IV, until 1877) recovered an old name, Oslo, only in 1925, after Thue's death.

Oslo). Two years later, he was elected to the Norwegian Academy of Science and Letters in Kristiania, married, became Professor of Mechanics in a technical college[7] in Trondheim and a member of the Trondheim Academy of Science and Letters. In 1903, he returned to the Royal Frederick University, this time as Professor of Applied Mechanics. From 1916 on, he was also associate editor of *Acta mathematica*, the journal created in 1882 by Gösta Mittag-Leffler as an international central organ in order to bring "Nordic" (that is, Scandinavian) mathematicians into the mainstream of mathematics.[8] During his life-time, Thue published 48 articles, about half of them between 1908 and 1912, see Figures 0.1 and 1.1; and despite the title of his chairs, and his life-long interest in physics, only 5 of these articles concern mechanics, against 9 in projective geometry and 21 in number theory.[9] He died on March 7, 1922, at the University Hospital in Kristiania.

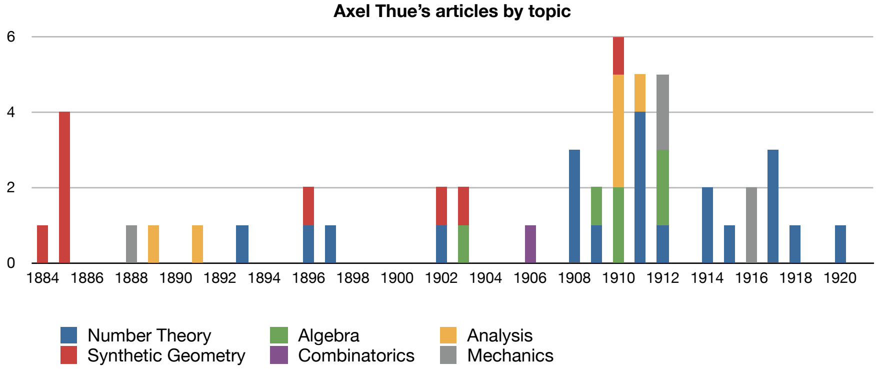


FIGURE 1.1.

Tønsberg, Thue's home town, is situated in the Oslofjord, about a hundred kilometers from Oslo. We possess a particularly evocative testimony about Tønsberg and its environment at the end of the nineteenth century, because the celebrated painter Edvard Munch, born only a few months after Thue, in December 1863, bought a summerhouse in Århgårdstrand, at ten kilometers north of Tønsberg. He painted in the region several of his masterpieces, including *Der Schrei der Natur* (The Scream), the *Melankoli* (Melancholy) series or, of course, *Fire piker i Åsgårdstrand*, Four Girls in

[7]Later becoming Trondheim Technical University.

[8]On this issue, see [49, 58]. Thue himself never published in *Acta mathematica*.

[9]The classification is mainly that of the *Jahrbuch über die Fortschritte der Mathematik*, which reviewed 40 different articles by Thue; I have retained only the main classification when several are proposed, and have added a classification for the 8 other articles by analogy (typically a particular case of a more general statement which did receive a review and thus a classification). For the sake of readability, in Figure 1.1, a handful of articles classified under different headings such as "series" or 'calculus" have been grouped together under the heading "analysis."

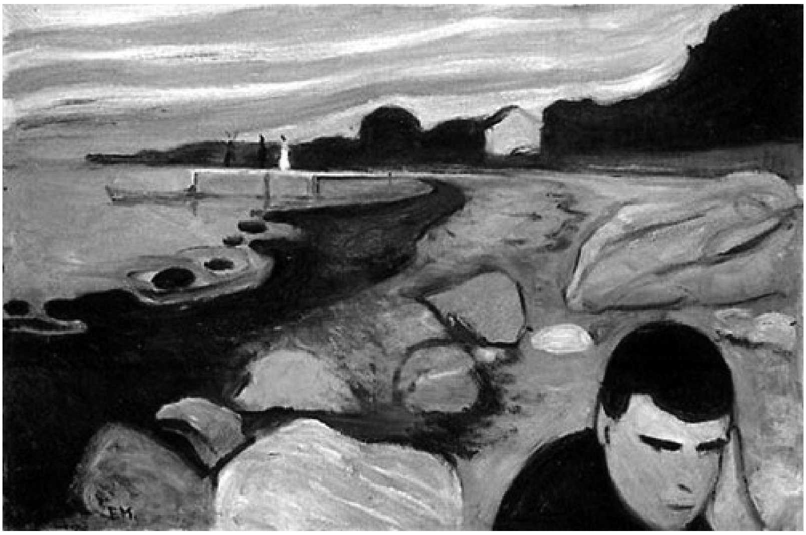


FIGURE 1.2. Melankoli, Edvar Munch, c.1892.

Åsgårdstrand. After a few years in Tønsberg, Thue was sent to Kristiana, more specifically to the *Aars og Voss Skole* (the Aars-Voss School). This private school was founded by two educational specialists, Jacob Jonathan Aars and Peter Voss, in the very year Thue was born, and, besides the classical curriculum, offered its students the *real* curriculum, that is, with an emphasis on mathematics and science instead of on Latin. By the end of the century, it was a very successful high school and a model in Norway, [1], [5]. Several important Norwegian scientists of Thue's generation were also students at the Aars-Voss School: the pole explorer and Nobel Peace Prize winner Fridtjof Nansen, born in 1861; the linguist Paul Boding, born in 1865 ; the physicist Kristian Birkeland, a specialist on Aurora Borealis, born in 1867, etc. And Thue's attachment to the Aars-Voss school can be traced down to at least 1912 when he contributed to the *Festskrift* of the school a paper on a theme dear to him, the principle of virtual velocities.

However, the most decisive asset of the school for Thue was a teacher, Elling Holst. In most biographies of Thue, Holst appears principally through an anecdote: he authored a booklet, *Om Poncelets betydning for Geometrien* ("On Poncelet's significance for geometries"), the title of which, mistyped or misread as *Pendelets betydning for Geometrien* ("On the significance of the pendulum for geometries") attracted the attention of the then physics-oriented Thue and finally turned him toward mathematics, [50, pp. 33–34]. The booklet received the Kronprinz Gold Medal in 1878. However, its author's influence on Thue's life extends far beyond a simple misprint.

FIGURE 1.3. Portrait of Elling Holst by Frederik Klem. ©Norsk Folkemuseum.

Elling Holst (1849–1915) is known through Norway because of his collections of Norvegian nursery rhymes, *Norsk billedbog for børn* (Norwegian picture books for children), published in 1888, 1890 and 1903. But he was first a trained mathematician, a student and protege of Sophus Lie; he studied with Felix Klein in Germany and took mathematical trips to Paris, Copenhagen, London. He defended a Ph D thesis on projective geometry (thus the title of the booklet) and first obtained a position in the Aars-Voss school, as well as a research stipend from the University of Kristiania; he was then appointed to a technical college and finally became Associate Professor at the University. Moreover, in 1886, Holst also launched a very influential mathematics seminar in Kristiania, apparently close to the German model.[10] Thue presented there about fifteen talks in the 1880s, in particular on the irrationality and the transcendence of e and π—unaware that Joseph Liouville had already discovered this extension, he even wrote in 1889 an article on the non-quadratic nature of e^2, following Liouville analogous result for e itself.[11]

According to A. Stubhaug, "[Elling Holst] was highly valued as a lecturer, and several famous mathematicians who began to study the subject at this time (Axel Thue, Kristian Birkeland, Carl Størmer, Richard Birkeland, etc), mention him later as an important source of inspiration."[12] Holst's own work, as well as his textbooks, was mostly centered on geometry. This is also the main topic of Thue's first mathematical writings, published in the 1880s. This orientation could also come from Lie's course on projective plane geometry, during the autumn 1884, for which Thue wrote up the lecture notes, [51, pp. 315-316], or more generally from the debates on the "new geometry" (including non-Euclidean geometry) in the preceding generation.

However, Holst also wrote a book on a quite different topic during this period: *Om høiere arithmetiske rækker. Samt nogle af de almindeligst forekommende konvergerende rækker med indledende sætninger om den hele*

[10]See the letter to Holst translated in [57, p. xx]. On the (official) organization of German seminars, see in particular [39, 40].

[11]As is well-known, Liouville gave in 1844 the first proof that there exist transcendental numbers, that is, numbers which are not solutions of an algebraic equation with integral coefficients, see subsection (3.1). New proofs of irrationality or non-quadraticity appeared as by-products of such work, since irrational (resp. non-quadratic) numbers are those which are not solutions of an algebraic equation of degree 1 (resp. degree 2) with integral coefficients. Charles Hermite's 1873 proof of the transcendence of e and the analogous result for π by Ferdinand von Lindemann in 1882 put these questions on the front stage in the 1880s.

[12]*Han blei høgt verdsett som førelesar, og fleire seinare kjende matematikarar (Axel Thue, Kristian Birkeland, Carl Størmer, Richard Birkeland m.fl.) som begynte å studere faget i denne tida, nemner han som ei viktig inspirasjonskjelde*, [52]. The biographical information here on Holst comes from this article. On Holst's relation to Lie, see [51, pp. 235-236, 240, 275-276, 330, 420-427]; on Holst's relation to Størmer, see [18, sec. 2.5]; on Holst's relation to K. Birkeland, see [17, p. 19-22]. Holst's pedagogical views are examined in [51, pp. 420-427].

funktion ("On higher arithmetical series as well as the most common converging series, with an introduction on the entire function"), [28]. Intended for advanced high school students,[13] the 56-page booklet followed the requirement of the school program, while seeking "to achieve the greatest possible rigor with the simplest possible means."[14] These constraints, and the small number of pages, make the content quite unusual indeed at the time: the difference between algebraic and transcendental functions was explained on page 3, with basic examples, the (Euclidean) division of polynomials on page 8, the method of indeterminate coefficients on pages 12 and 13, binomial coefficients on page 20. The fourth part, devoted to arithmetical series, introduced, in particular, finite differences to compute the sums of the values of arithmetical functions on integers (f.i. the sum of the cubes, $\sum_0^n x^3 = [\frac{n(n+1)}{1\cdot 2}]^2$, page 28). At the end, Holst provided criteria for a series to converge and applied them to a variety of examples, both numerical ("to compute $\sqrt[5]{8}$ with 5 decimals") and theoretical (the study of the exponential function).

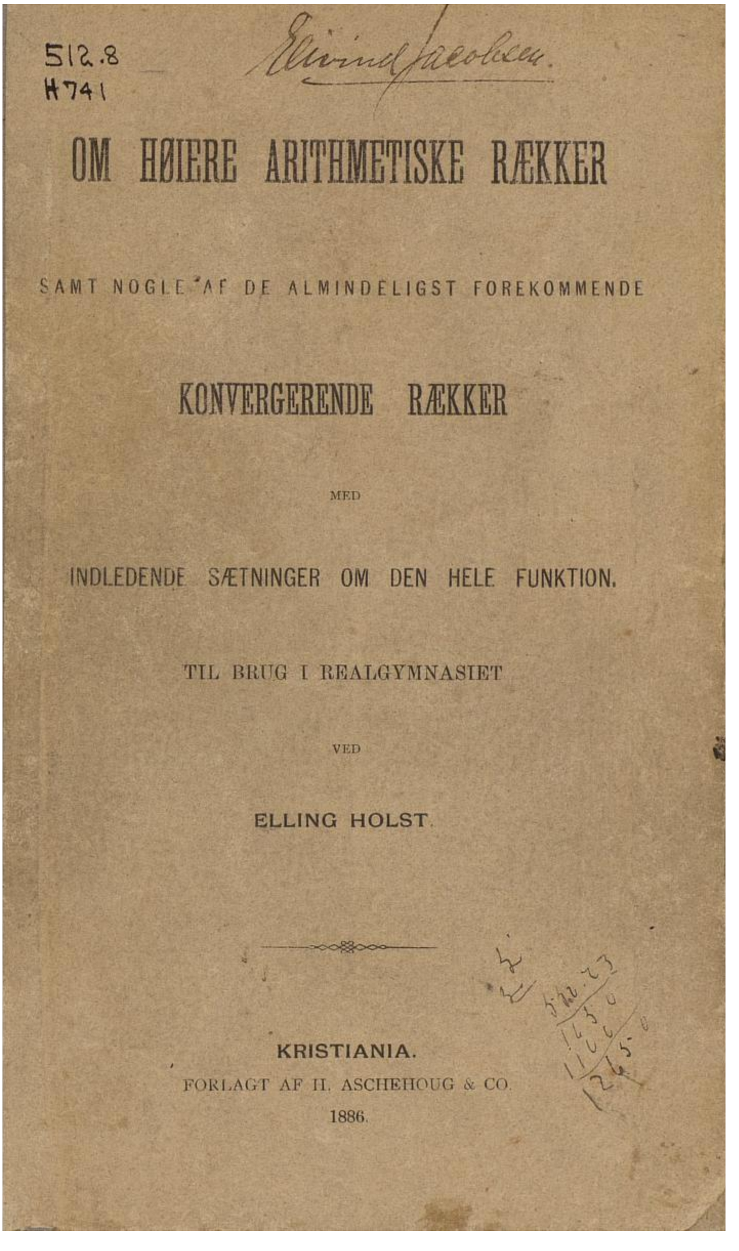
Several features of the book are quite remarkable. Analysis appears here mostly as the domain that deals with infinite processes (much more than, say, continuity) as they branch off directly from arithmetic and algebra: operations on integers are transposed into analogous operations on polynomials; numbers, polynomials, series are steps in a common approach; reciprocally, as the title shows, arithmetical series are at the core of the study, as objects joining arithmetic, algebra and analysis. Moreover, compared to standard elementary textbooks on analysis, geometrical representations are totally missing: there is no discussion of the relation between curves and functions, nor any graphical discussion of functions.[15]

It is now instructive to compare the contents of this booklet to Thue's articles. To give one example, all the more interesting because it comes from a slightly later date and touches upon Thue's new, independent research

[13]The school program was defined through certain portions of specific textbooks. Holst's book completed the 1860 *Lærebog i arithmetik og algebraens elementer* ("Textbook in arithmetik and algebra elements") by Ole Jacob Broch, which began with the addition of integers and ended with logarithms and sequences. Broch, an influential figure of Norwegian mathematics and politics, described his textbook as preparing for the *examen artium* (an equivalent of the French *baccalauréat*, this exam, prepared in high schools, was compulsory for entrance at the university) and for the *examen philosophicum* (propedeutic years at the university). According to [1, p. 172], Holst's book was an important part of the curriculum of the last year of high school, in the *real*, that is scientific, section, at the beginning of the twentieth century. On the educational system and its stakes in nineteenth-century Norway, see [1, 5], and [51, ch. IV, V].

[14]*at opnaa størst mulig stringens ved de simplest mulige midler*, [28, Forord].

[15]Progressions and logarithms, mostly in the perspective of computations, were a classical extension of elementary algebra in high school, but not arithmetical functions nor finite differences. To appreciate the contrast with other European textbooks in the teaching of analysis, see [41]; for the specific case of France, the high school programs are given in [3].



OM HØIERE ARITHMETISKE RÆKKER

SAMT NOGLE AF DE ALMINDELIGST FOREKOMMENDE

KONVERGERENDE RÆKKER

MED

INDLEDENDE SÆTNINGER OM DEN HELE FUNKTION.

TIL BRUG I REALGYMNASIET

VED

ELLING HOLST.

KRISTIANIA.

FORLAGT AF H. ASCHEHOUG & CO.

1886.

FIGURE 1.4. *Om høiere arithmetiske rækker*, by Elling Holst

interests, namely number theory, let us look at Thue's "Bevis for Fermats og Wilsons sætninger," ("proof of Fermat's and Wilson's theorems") published in 1893, [53]. Noting that the binomial coefficient $\binom{n}{p}$, for n a prime number and p an integer, $n > p$, is divisible by n, Thue first deduces that, for any integer a

$$\begin{array}{rcl} a^n - (a-1)^n & = & nk+1 \\ (a-1)^n - (a-2)^n & = & nh+1 \\ \cdots & \cdots & \cdots \\ (a-(a-1))^n - (a-a)^n & = & 1 \end{array}$$

for certain integers k, h, etc. By adding all the lines, he then obtains : $a^n = nR + a$, for a certain integer R. This shows that n should divide $a^n - a$, thus $a^{n-1} - 1$ if n and a are relatively prime; this is Fermat's Little Theorem.

As for Wilson's theorem, Thue uses the successive finite differences of the polynomial $F(x) = x^n$:

$$\Delta^1 F(x) = F(x+1) - F(x) = nx^{n-1} + \cdots ,$$

$$\Delta^2 F(x) = \Delta^1 F(x+1) - \Delta^1 F(x) = n(n-1)x^{n-2} + \cdots , \cdots , \Delta^n F(x) = n!$$

A linear combination of these equalities, for $n = p-1$ and $x = 0$, provides Thue with the equation:

$$-\Delta^{p-1} F(0) = 1 - (-\Delta^1 F(1) + \Delta^2 F(1) - \cdots - \Delta^{p-2} F(1)).$$

Fermat's Little Theorem then allows him to conclude that $(p-1)! + 1$ is a multiple of p, which is Wilson's theorem.

Elementary as they are, these proofs clearly display a proximity to Holst's privileged tools, something that we find again in other writings of Thue. Polynomials and (formal) series are put at the core of number-theoretical proofs, as well as their paraphernalia: finite differences, binomial coefficients, multiplication of series, etc.

2. *Wanderjahre*: The German trip

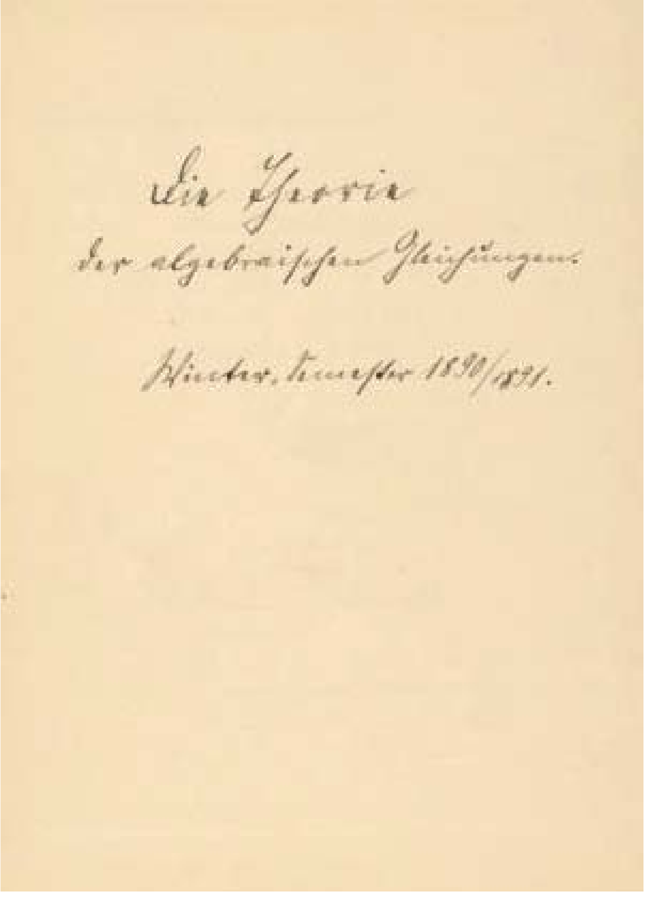
Elling Holst had remained in close contact with Sophus Lie when the latter left Norway for Germany in 1886. This link operated in particular to provide young promising Norwegian scientists with an opportunity to travel to Germany and to complete—or even acquire—there their scientific culture. Besides Thue, the meteorologist Vilhelm Bjerknes, the future specialists in insurance mathematics Alf Guldberg and Arnfinn Palmstrøm, the educator and politician Anton Alexander, etc., studied science abroad during the 1890s. Where exactly to go in Germany was a matter of detailed discussion; while Berlin and Göttingen were the two obvious choices at the time, Bjerknes went to Bonn for several years to become the assistant of Heinrich Hertz, and Leipzig was promoted by Lie as soon as he obtained Felix Klein's former position there. Concerning Thue in particular, Lie wrote to Holst: "I have advised Axel Thue to apply for a traveling scholarship this winter. Thue has shown in a series of original investigations that he has many of the qualifications needed to become an outstanding mathematician. [...]. Yet, I still have a definite impression that his mathematical knowledge does not do justice to his gifts and his enthusiasm. It is for that reason that I consider it to be absolutely necessary for him to go abroad. [...] I am applying to you in this matter because Thue is first and foremost your pupil, and I am certain that you nourish a lively interest in his development as a mathematician. I therefore beg you to do all you can to obtain a traveling scholarship for Thue immediately," [57, p. xvii]. He added: "I would be reluctant to send him to Göttingen. [...] Guldberg and Bjerknes are there, on my advice. It would be too unequal if Thue were to go there too. In Berlin, according to all testimony, there is not much to do. On several occasions, young mathematicians have left Berlin and come to me," [51, p. 399].

Thue thus went to Leipzig in the summer of 1889 with a grant, but the stay was not very fruitful: after a strong start with private lessons from Lie, the latter left Leipzig for several weeks, during which time Thue fell severely ill and could not work any more. Then Lie became depressed and had to leave. This certainly justified what is to be found in Thue's biographies: while the Leipzig stay at first stimulated Thue's own research, he did not receive any decisive influence from Lie (especially because what Lie wanted were co-workers on his own program). However, Thue obtained a grant for a second German stay, from the spring of 1890 to the summer of 1891, and this time went to Berlin. There he attended a series of courses from Leopold Kronecker, Lazarus Fuchs, Hermann von Helmholtz and Georg Hettner, and, moreover, participated in the animated social life of Berlin mathematicians.[16] Helmholtz's course, more than the others, seems to have captured Thue's imagination, because of its connection with infinity and the problem of space. However, it is worth noticing that some of the other themes discussed in Berlin were close to Thue's former and later work: Hettner offered, for instance, a course on e and π, that is, on the various proofs of their transcendence developed in the wake of Hermite's and Lindemann's achievements.[17]

As for Kronecker, he gave his last course during the autumn of 1891—he died at the end of December—when Thue had just returned to Norway. But Thue attended two courses given by Kronecker. On June 20, he explained to Holst: "During the first semester, [Palmstrøm] and I attended Kronecker's lectures on the theory of algebraic equations. He is remarkable for his great depth and thoroughness, but has the bad habit in the fire of his enthusiasm

[16] A letter from Thue to Holst, translated into English in [57, p. xix-xx], provides some vivid comments on these courses, as well as on a ball at Fuchs's home. Thue also attended the mathematical seminar of the University and gave some talks at the meetings of the Berlin Mathematical Society.

[17] We recall that Thue has explained the topic in Holst's mathematical seminar and written a small article in connection to this; he was not particularly impressed by Hettner's course, nor by Fuchs's one for that matter,[57, p.xx].



Die Theorie
der algebraischen Gleichungen.
Winter-Semester 1890/1891.

FIGURE 2.1. Title page of L. Kronecker's course. Fonds Hensel, Bibliothèques de l'université de Strasbourg.

to relate definitions and other absolutely essential information altogether too rapidly. One can thus too easily miss the point. He is an extremely likeable man, but prefers to do all the talking himself, and at considerable length. [...] Kronecker gives quite extraordinarily lively lectures and he understands how to make things palatable and one has a strong recurrent feeling that he enjoys what he serves up."[18]

What Kronecker served to Thue and his other students during the Wintersemester 1890–1891 is known in detail through the carefully handwritten

[18] *I det förste semester hörte vi P. og jeg Kroneckers forelæsninger over de algebraiske ligningers theori. Han udmarker sig ved stor dybde og grundighed, men har den slemme vane i begeistringens ild at udtale definitioner og andre absolut nödvendige oplysninger altfor fort. Man gaar jo da so let glip af meningen. Han er en saare elskværdig mand, men vil helst tale selv og da lange. [...] Kronecker har et ganske overordentlig livligt foredrag og forstaar at gjöre tingen smagelig og man föler at han selv nyder, hvad han serverer.*

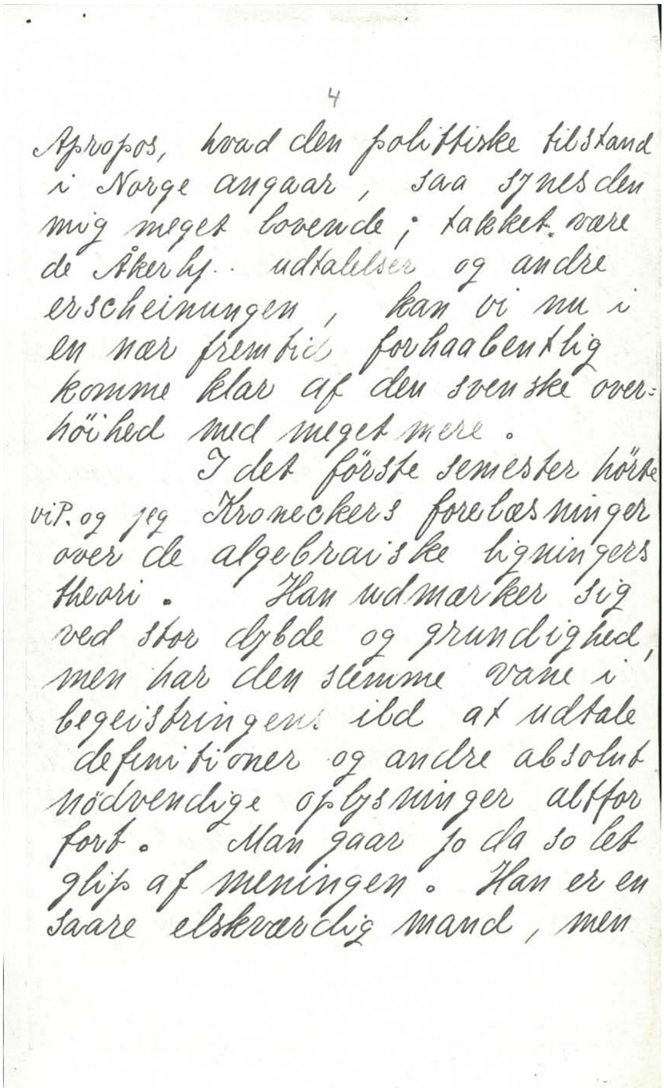
lecture notes kept in the Fonds Kurt Hensel of the University of Strasbourg which digitized them in 2012,[29].[19] In this case, we do not know the name of the writer (usually a student). The manuscript is composed of 420 pages, to which are added a table of contents and a few notes, totaling 666 digitized pages. After a first lecture on the history of algebra following Kronecker, the course includes treatments of quadratic and cubic equations, Sturm's theorem on the number and localization of real roots of equations, Lagrange's interpolation formula, resultants of polynomials, the fundamental theorem of algebra, Gauss's and Abel's works on, respectively, cyclotomic equations and solvable equations and a taste of Galois theory, all this presented from Kronecker's own point of view.[20]

Because of his well-known attacks on Georg Cantor on the one hand and Karl Weierstrass on the other, Kronecker is usually presented as a (dictatorial) intuitionist *avant la lettre*. His claims that all mathematics should be built only upon natural numbers through finite procedures are much more commented on than his actual mathematical work—in which elliptic functions and algebraic equations are blended with number theory. However, recent historiography has articulated Kronecker's priorities and his mathematical work in much more detail: it has shown how Kronecker aimed at taking "refuge in the safe haven of actual mathematics," in his own words, through an emphasis on concrete computations and formulas.[21] Although there is no way we can speak of a Kroneckerian influence on Thue, in the sense that Thue would have adopted Kronecker's program or framework, it is still interesting to understand to what kind of mathematics Thue was exposed. One aspect is the role of approximation, including classical results on real roots. For Kronecker, approximation was a major link between theoretical mathematics and science at large, a conception of which Thue, motivated as he was by questions in physics and technology, would probably not have been ignorant. It is also coherent with the proofs Thue had studied concerning irrationality and transcendence; continued fractions and their algebraic analogues appear there as key objects and these tools—which were also ways of thinking about the problems—were explained at length in Kronecker's course. Another aspect is the emphasis on polynomials. Kronecker's project was to subsume arithmetic, algebra and analysis itself into a higher "generalized arithmetic": this arithmetic operates in rationality domains, that is quotients of the ring of polynomials in several indeterminates $\mathbf{Z}[X_1, X_2, \cdots, X_n]$ and the study of algebraic numbers, that

[19]We do not have the notes of the 1891 lecture on definite integrals, but older lectures from Kronecker on the same subject are also available, which I shall not discuss here.

[20]On the variegated development of Galois theory after Galois, see [19]. On Kronecker's perspective on Galois theory, see [16].

[21]See [16, p.11]. Harold Edwards has devoted several articles to a detailed analysis of Kronecker's algorithmic perspective, see [12, 13, 14, 16]. On Kronecker's conception of numbers, in particular his last lectures, see [7, 6].



4

Apropos, hvad den politiske tilstand i Norge angaar, saa synes den mig meget lovende; takket være de Åkerhj.. udtalelser og andre erscheinungen, kan vi nu i en nær fremtid forhaabentlig komme klar af den svenske overhöihed med meget mere.

I det förste semester hörte vi P. og jeg Kroneckers forelæsninger over de algebraiske ligningers theori. Han udmærker sig ved stor dybde og grundighed, men har den slemme vane i begeistringens ild at udtale definitioner og andre absolut nödvendige oplysninger altfor fort. Man gaar jo da so let glip af meningen. Han er en saare elskværdig mand, men

FIGURE 2.2. Extract of a letter from A. Thue to E. Holst, June 20, 1891, Archives Axel Thue. With the kind permission of NTNU University Library, Dora Library, Trondheim.

is, roots of polynomials, is done via rationality domains modulo such polynomials. The “safe haven” of Kronecker, as illustrated in his 1890 course as well as in his important 1882 paper “Grundzüge einer arithmetischen

Theorie der algebraischen Grössen," was based on extensive computations on polynomials.[22]

As explained above, Thue returned to Norway during the summer of 1891 and spent a few years as Assistant Professor in Kristiania, before leaving for a professorship in the Technical College of Trondheim, where he remained until 1903. Thue described his years in Trondheim as unpleasant and spent in mathematical isolation; he published very little during this time, but it gave him the opportunity to develop his own project on approximation and algebraic numbers, and perhaps most of the material he published after his recruitment as full professor in Kristiania, on Diophantine equations, rational approximation or sequences of signs, [57, p. xxi]. I shall now focus on his celebrated work on diophantine approximation, and, in particular, on the features in it which illustrate Thue's proximity with some of the mathematics just described.

3. Thue's paper on the approximation of algebraic numbers

Thue's paper, "Über Annäherungswerte algebraischer Zahlen" ("On approximate values of algebraic numbers") was published in 1909 in the Berlin *Journal für die reine und angewandte Mathematik*, known as *Crelle's Journal*, although the editor-in-chief was by then Kurt Hensel. Without any introduction or contextualization of any kind, it began with the statement of a main theorem, "Theorem I":

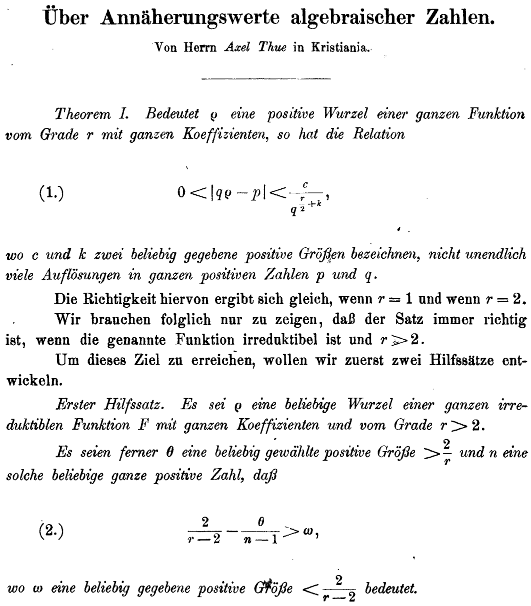
Theorem 1. *Let ρ be a positive root of an entire function of degree r with integral coefficients. The relation*

$$0 <| q\rho - p |< \frac{c}{q^{\frac{r}{2}+k}}$$

where c and k are any given positive quantities, does not have infinitely many solutions in positive integers p, q.

The paper then contained a 19 page proof, based on two lemmas, and three different applications. A crucial point of Theorem I is the power $\frac{r}{2}$, half of the degree of the algebraic number ρ—or $\frac{r}{2}+1$, if one divides the two sides by q, according to what is now the standard presentation. As it is well-known, this power was improved as $2\sqrt{r}$ by the work of Siegel, revisiting Thue's paper in [44, 45], then by several other authors, before Klaus Roth's better (in fact best) result in 1955, with 1 instead of $\frac{r}{2}$: for any positive real number k, $0 <| \rho - \frac{p}{q} |< \frac{1}{q^{2+k}}$ has only finitely many solutions (p, q) with relatively prime integers p and q.

[22]See [12, p. 133–134], for instance, for an illuminating explanation of Kronecker's concept of a splitting field, and [15] for a convincing attempt to develop this "haven" in a constructive, Kroneckerian manner.



Über Annäherungswerte algebraischer Zahlen.

Von Herrn *Axel Thue* in Kristiania.

Theorem I. Bedeutet ϱ eine positive Wurzel einer ganzen Funktion vom Grade r mit ganzen Koeffizienten, so hat die Relation

$$(1.)\qquad 0 < |q\varrho - p| < \frac{c}{q^{\frac{r}{2}+k}},$$

wo c und k zwei beliebig gegebene positive Größen bezeichnen, nicht unendlich viele Auflösungen in ganzen positiven Zahlen p und q.

Die Richtigkeit hiervon ergibt sich gleich, wenn $r = 1$ und wenn $r = 2$.

Wir brauchen folglich nur zu zeigen, daß der Satz immer richtig ist, wenn die genannte Funktion irreduktibel ist und $r > 2$.

Um dieses Ziel zu erreichen, wollen wir zuerst zwei Hilfssätze entwickeln.

Erster Hilfssatz. Es sei ϱ eine beliebige Wurzel einer ganzen irreduktiblen Funktion F mit ganzen Koeffizienten und vom Grade $r > 2$.

Es seien ferner θ eine beliebig gewählte positive Größe $> \frac{2}{r}$ und n eine solche beliebige ganze positive Zahl, daß

$$(2.)\qquad \frac{2}{r-2} - \frac{\theta}{n-1} > \omega,$$

wo ω eine beliebig gegebene positive Größe $< \frac{2}{r-2}$ bedeutet.

FIGURE 3.1. First page of Thue's 1909 article, displaying Theorem I

3.1. Approximation before Thue. In his articles, Siegel conjectured what would be the best result to expect. But he also launched what would be the standard chronology of Thue-type results, more or less reduced before Thue's paper to Liouville's result of 1844.[23] Approximation issues were in fact rather prominent in certain number-theoretical circles at the end of the nineteenth century, and it is useful to briefly survey those which Thue knew well, in order to better understand some techniques and ideas readily available to him.

Liouville's communications to the French Academy of Sciences in 1844, [32, 33], which he reproduced in a 1851 article in his own *Journal de mathématiques pures et appliquées*, [34], contain the oldest known proof of transcendence. Liouville's argument relies on the continued fraction expansion of a real number, ρ,

[23]See for instance Edmund Landau's *Vorlesungen*, [30, vol. 3, pp. 37–65].

$$\rho = [a_0, a_1, \ldots, a_k, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

and on its main properties. The reduced fractions $\frac{p_k}{q_k} = [a_0, a_1, \ldots, a_k]$ give the best rational approximations to the real number ρ. One has

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}.$$

Moreover if ρ_k is the complete quotient of order k, that is, the real number satisfying $\rho = [a_0, a_1, \ldots, a_{k-1}, \rho_k]$, then

$$\rho = \frac{p_k \rho_k + p_{k-1}}{q_k \rho_k + q_{k-1}}$$

from which one deduces that

$$\mid \rho - \frac{p_k}{q_k} \mid = \frac{1}{q_k(q_k \rho_k + q_{k-1})}. \tag{3.1}$$

As the q_k and the ρ_k are greater than 1, $q_k \rho_k + q_{k-1}$ is greater than q_k and thus Equation (3.1) shows that for every real number ρ, there exists infinitely many rational numbers $\frac{p}{q}$ such that $\mid \rho q - p \mid < \frac{1}{q}$, or, $\mid \rho - \frac{p}{q} \mid < \frac{1}{q^2}$.

Now let ρ be an algebraic number, that is, the root of an algebraic equation f of degree r with integral coefficients; one can assume that f is irreducible, in particular has no rational roots. Then

$$f(x) = a(x - \rho)(x - x_1) \cdots (x - x_{r-1}),$$

where the x_i are the other roots of f. In his 1844 papers, Liouville remarks that, for any rational number $\frac{p}{q}$,

$$\frac{f(\frac{p}{q})}{\frac{p}{q} - \rho} = a(\frac{p}{q} - x_1)(\frac{p}{q} - x_2) \cdots (\frac{p}{q} - x_{r-1}). \tag{3.2}$$

On the one hand, $\mid q^r f(\frac{p}{q}) \mid$ is an integer, which is not 0 because f has no rational root, and thus is equal to or greater than 1. On the other hand, when the fractions $\frac{p}{q}$ are successively the reduced fractions $\frac{p_k}{q_k}$ of the continued fraction expansion of ρ, they approximate ρ, thus are closer and closer to ρ and the differences $\mid \frac{p_k}{q_k} - x_i \mid$ are bounded. Liouville deduced from Equation (3.2) that there is a real C such that $\mid \rho - \frac{p_k}{q_k} \mid \geq \frac{C}{q_k^r}$. Equation (3.1) thus provides him with an upper bound $C' q_k^{k-2}$ for the ρ_k, and *a fortiori* for their integral parts, which are the coefficients a_k of the continued fraction expansion of ρ. Liouville then explained how to construct a transcendental number by allowing the coefficients a_k of its continued fraction expansion to increase beyond this bound.[24]

Although Liouville's article contains indeed a Thue-like result, with the exponent r instead of $\frac{r}{2}+1$, it is perhaps not the only point of departure to contextualize Thue's work. Two years earlier, on April 14, 1842, Peter Gustav Lejeune-Dirichlet presented to the Berlin Academy of Sciences some results on the values of linear forms for integers. He began with the remark that the theory of continued fractions shows that for any real number ρ, and for infinitely many integers p, q, the value at $x = p, y = q$ of the linear form $x - \rho y$ is less than $\frac{1}{y}$ (this is Equation (3.1) above). The aim of his communication was to generalize this statement to linear forms with real coefficients in any finite number of variables: instead of the argument based on continued fractions, he used what is now called the pigeonhole principle.[25] Then, Dirichlet applied it to show that certain families of Diophantine equations have infinitely many solutions—like the theory of continued fractions was used to solve Pell-Fermat equations.

Such results were well known a few years later to the young Charles Hermite—close to Liouville and very well informed about German mathematics— when he reinvested the issue of rational approximation through quadratic forms. For a real number A, Hermite introduced a family of positive definite binary quadratic forms f_Δ, indexed by real positive numbers Δ, such that

$$f_\Delta(x, y) = (x - Ay)^2 + \frac{y^2}{\Delta}.$$

According to classical results on forms on integral values, there exist in this case two integers m and n such that

$$f_\Delta(x, y) \leq \frac{2}{\sqrt{3}\sqrt{\Delta}},$$

from which Hermite easily deduced that $\mid A - \frac{m}{n} \mid \leq \frac{c}{n^2}$, for a constant c. He also explained the law of formation of the integers (m, n) when Δ increases from 0 to ∞, in connection with the development as a continued fraction of the real number A. Hermite also extended his result to simultaneous

[24] A variant relying on what Liouville calls a theorem of Lagrange—the intermediate value theorem—is given in the first note, [32]. The case where $r = 1$, and hence the continued fraction of ρ is finite is discussed in [34], as well as the extension of the procedure to complex numbers.

[25] See [10]. The principle is that if one puts a collection of strictly more than M discrete objects in M holes or boxes, two objects at least will share the same hole. Dirichlet used it here (without any specific name) to distribute certain fractions in intervals of a given length. Although such arguments had appeared earlier, it is from Dirichlet's papers that the explicit use of the pigeonhole principle in number theory, in particular as a substitute for continued fractions, developed in the nineteenth century.

rational approximations of a finite number of real numbers, again via the study of minima of appropriate quadratic forms.[26]

However, Hermite later provided another and more celebrated generalization of his statement on approximation: his proof of the transcendence of e. Hermite's 1873 series of notes in the *Comptes rendus de l'Académie des sciences*, "On the exponential function," indeed began exactly where his work on rational approximation had ended: instead of generalizing the theory of (ordinary) continued fractions "one can propose," he writes, "a similar generalization of the theory of algebraic continued fractions, while searching for approximate expressions of functions by rational fractions, in such a way that the series expansions in increasing powers of the variable coincide up to a fixed power, " [26, vol. 3, p. 150].

Hermite's original proof indeed relied upon transposition of ideas and expressions from rational numbers to rational functions, considered as series. Moreover, Hermite's works taken as a whole stressed a path for approximation going from numbers (that is, zero-degree polynomials!) to first-degree polynomials to general polynomials, and finally to series.[27] Here again, polynomials, with concrete, often very long computations, were at the core of Hermite's mathematical activities, in particular with respect to approximation issues.

That Thue, back in Norway, was still involved in such issues, as he was before, is well documented. In a letter to Holst from August 1902, Thue wrote for instance: "Nevertheless I have in my deathly solitude, where no one is interested in my stuff, produced significantly even more and better things than in the past. Thus I have developed a theory, accordingly to which I partly arrive at Hermite's and Lindemann's results for e and π. The reason why I have not published the stuff I have in stock partly lies in the fact that I may push my investigations further forward and would like first see a proper conclusion to them."[28]

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[26] Hermite elaborated his results for several real numbers from 1847 onwards, in a series of letters to Carl Gustav Jacob Jacobi, which were only published in 1850, [26, vol. 1, p. 106]. For two numbers A and B, for instance, he used the quadratic ternary form $(x-Az)^2+(y-Bz)^2+\frac{z^2}{\Delta}$ and obtained rational numbers $\frac{m'}{m}, \frac{m''}{m}$, such that both $\mid \frac{m'}{m}-A \mid$ and $\mid \frac{m''}{m}-B \mid$ are less than $\frac{2\sqrt{2}}{\sqrt[4]{27}}\frac{1}{|m\sqrt{|m|}|}$, that is, simultanous approximations of A and B by rational numbers with the same denominator m and an error of order less than $m^{\frac{3}{2}}$. The existence of minima of binary and ternary quadratic forms (with integral coefficients) on integral values is implicit in Carl Friedrich Gauss's *Disquisitiones arithmeticae*, but Hermite extended Gauss's proof to forms with real coefficients and any finite number of variables, see [22]. The case of a single number A, mentioned in the third letter, [26, vol. 1, p. 140], is the one put forward by Émile Picard in his preface to Hermite's works, [26, vol. 1, p. xi].

[27] We point out, in this direction, Hermite's proof of the irrationality of e which can be described as a first-degree version of his transcendence proof of 1873, see [26, vol. 3, p. 154].

[28] This is quoted in [43, p. 143]: *Ikke desto mindre har jeg i min drepende ensomhet, hvor ingen interesserer seg for mine ting, likevel produsert betydelig mer og bedre ting enn før i tiden. Jeg har således utviklet en teori, etter hvilken jeg blant annet kommer til* Hermites *og* Lindemanns

3.2. The first lemma: replacing numbers by polynomials. Thue's proof relied on two lemmas. The first one begins with a *modus operandi* presenting a striking analogy with what we have met above. Thue replaces the expression he is studying, $\mid q\rho - p \mid$, with ρ an algebraic number of degree $r > 2$ and p, q integers, by the same expression, but with polynomials with integral coefficients in the place of integers: $\mid Q(x)\rho - P(x) \mid$. He works then in $\mathbb{Z}[x, \rho]$, adjusting P and Q so that the expression provides a high-degree factor $(\rho - x)^n$. That is, he explains first how to provide approximations by rational functions, with a control on the degrees of their numerator and denominator polynomials and the size of their coefficients. More precisely:

Lemma 1. *Let ρ be a root of an irreducible polynomial F with integral coefficients and degree $r > 2$. Also let θ be an arbitrary chosen quantity $> \frac{2}{r}$ and n a positive integer such that*

$$\frac{2}{r-2} - \frac{\theta}{n-1} > \omega$$

where ω is an arbitrary positive magnitude $< \frac{2}{r-2}$. Finally, let m be the positive integer that satisfies the relation:

$$m \leq (\frac{r-2}{2} + \frac{1}{\theta})(n-1) < m+1. \tag{3.3}$$

Then one can always find polynomials $f(x)$, $P(x)$ and $Q(x)$, with integral coefficients and independent of the choice of the root ρ, and also positive magnitudes S and T, depending only on F, θ and ω, such that

$$\rho Q(x) - P(x) = (\rho - x)^n \cdot [f_1(x)\rho^{r-1} + f_2(x)\rho^{r-2} + \cdots + f_{r-1}(x)\rho + f_r(x)], \tag{3.4}$$

where the degree of each f is not greater than m, the degree of P and Q not greater than $m+n$, the absolute value of each coefficient of the functions f is smaller than T^n, and the absolute value of each coefficient of P and Q is smaller than S^n.

We notice that, when θ and ω, chosen arbitrarily as indicated, are determined, the conditions only impose a minoration on n; the lemma thus provides a family of rational functions $\frac{P(x)}{Q(x)}$ indexed by n. Reciprocally, however, for n determined, one can always construct polynomials satisfying Equation (3.4); in particular, for $n > 1$, one can find adequate θ and ω. Of course, the trivial identity, $\rho - x = (\rho - x) \cdot 1$, will correspond to $P(x) = x$, $Q(x) = 1$, $f_1(x) = f_{r-1}(x) = 0$, $f_r(x) = 1$ for $n = 1$.

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resultater om e og π. Grunnen til at jeg ikke har offentliggjort de ting jeg har liggende på lager, ligger dels deri, at jeg kan føre undersøkelsene lengre frem og gjerne først vil se en ordentlig avslutning på dem. Thue added also that he was not sure how new his resultats were, which made him uncomfortable and paralyzed his imagination.

3.3. An example : $F(x) = x^3 - ax - b$, a, b rational numbers, F irreducible. One year before the publication of his 1909 paper in *Crelle*, Thue published explicit computations for a family of examples, [54] (other examples are handled in his papers of this period): the case where ρ is a root of an irreducible cubic polynomial $F(x) = x^3 - ax - b$. In this case, he proved that for any positive integer I, there exist polynomials $P_I(x)$ and $Q_I(x)$, of degrees less than $3I + 1$, and $R_I(x)$, of degree less than I, such that

$$\rho P_I(x) - Q_I(x) = (x - \rho)^{2I+1} R_I(x).$$

In other words, such polynomials satisfy Equation (3.4), for $r = 3$, $n = 2I + 1$ and $m = I$.

To give a concrete flavor to the lemma, here are the polynomials given by Thue in the case $I = 2$ (thus $n = 5$).

$$\begin{aligned}
P_2(x) &= 81abx^7 + (378b^2 + 70a^3)x^6 + 567a^2bx^5 + (945ab^2 + 70a^4)x^4 \\
&\quad + (175a^3b + 945b^3)x^3 + (378a^2b^2 - 14a^5)x^2 + (189ab^3 - 7a^4b)x \\
&\quad + (2a^6 - 29a^3b^2 + 135b^4) \\
Q_2(x) &= (16a^3 + 135b^2)x^7 + 378a^2bx^6 + (945ab^2 + 112a^4)x^5 + \\
&\quad + (945b^3 + 490a^3b)x^4 + 945a^2b^2x^3 + (756ab^3 + 14a^4b)x^2 + \\
&\quad + (378b^4 + 7a^3b^2)x + (27a^2b^3 - 2a^5b) \\
R_2(x) &= [405abx + 110a^3 - 135b^2]\rho^2 + [81abx^2 - (10a^3 + 297b^2)x \\
&\quad - 108a^2b]\rho - [(16a^3 + 135b^2)x^2 + 378a^2bx + (112a^4 - 270ab^2)].
\end{aligned}$$

As required in the statement of the lemma 1, the degree of the polynomials $P_2(x)$ and $Q_2(x)$ is equal to $m + n$, here 7; that of f_1, f_2, f_3 is less than or equal to $m = 2$.

3.4. Proof of lemma 1 (sketch). For reasons of space, we shall only explain the main steps of Thue's original proof of the lemma.[29]

First of all, for $i = 0, \cdots, r-1$, Thue develops $\rho^i(\rho - x)^n$ in $\mathbb{Z}[x][\rho]$, using of course the fact that $F(\rho) = 0$, and thus that ρ^r (and the higher powers of ρ) are linear combinations with integral coefficients of $1, \rho, \cdots, \rho^{r-1}$. That is, he proves, that for $i = 0, \cdots, r-1$, and $j = 1, \cdots r-1$, there exist polynomials $B_j^{(i)}$ of degree n, with integral coefficients bounded by a

[29] As mentioned above, Thue gave concrete constructions of the polynomials of Lemma 1 for specific cases in various papers; he also devoted a whole paper to this issue, [56].

quantity T_0^n, where T_0 depends only on the coefficients of F, such that

$$(3.5)\quad \rho^i(\rho - x)^n = B_1^{(i)}(x)\rho^{r-1} + B_2^{(i)}(x)\rho^{r-2} + \cdots + B_{r-1}^{(i)}(x)\rho + B_r^{(i)}(x).$$

Thue considers then general expressions analogous to those in the right-hand part of Equation (3.5) above; that is, polynomials $U(x)$ such that

$$U(x) = C_1(x)\rho^{r-1} + \cdots + C_{r-1}(x)\rho + C_r(x)$$

where the polynomials C are of degree $\leq m$ and have integral coefficients bounded by an integer s. He notes that there are finitely many: each coefficient of the $C_i(x)$ lies between $-s$ and s, thus can take only a finite number of values, and there is a finite number, $(m+1)r$, of such coefficients, thus only $M = (2s+1)^{(m+1)r}$ possible polynomials.

Now, the (finitely many) expressions

$$(\rho - x)^n U(x) = G_1(x)\rho^{r-1} + G_2(x)\rho^{r-2} + \cdots + G_r(x)$$

are such that the degrees of the G are less than $n+m$ and their coefficients are bounded by a quantity N.

What is wanted to get (3.4) is to insure that in one of these expressions $(\rho - x)^n U(x)$ at least, no terms in $\rho^{r-1}, \cdots, \rho^3, \rho^2$ appear.

To do this, Thue applies the pigeonhole principle. He cuts the interval [-N, N] into h subintervals of length $\frac{2N}{h}$. If

$$h^{(m+n+1)(r-2)} < M = (2s+1)^{(m+1)r},$$

at least two polynomials, U_1 and U_2, provide corresponding G_1, G_2, $\ldots$, G_{r-2}, whose coefficients lie in the same interval.

Then, if one also chooses $h > 2N$, the coefficients of ρ^r, ρ^{r-1}, $\ldots$, ρ^2 in the expression of $(\rho - x)^n(U_1 - U_2)$ vanish, which concludes the proof of (3.4), with $U = U_1 - U_2$. It is to be noted that Thue devoted several papers to variants, explicit versions and generalizations of his first Lemma; he also announced in a footnote, [55, p. 292] that the lemma can be extended to polynomials with algebraic coefficients, as well as to polynomials in several variables.

3.5. The second lemma. The second step in Thue's proof is to construct two distinct rational approximations of ρ, with bounded denominator. More precisely,

Lemma 2. *Let F, ρ, r, ω, n be as in the first lemma. We now assume that ρ is real, $\theta > 2$ and that p and q are two integers such that $q > 0$ and $\mid \rho q - p \mid < 1$. One can then always find integers A_0, B_0, A_1, B_1 such that*

$$\frac{A_0}{B_0} \neq \frac{A_1}{B_1}$$

$$\mid \rho B_i - A_i \mid < \left| \left[(\rho q - p)^{(1-\frac{2}{\theta})} C q^{(\frac{r-2}{2}+\frac{1}{\theta})} \right]^{n-1} \right|$$

$$\mid B_i \mid < D^{n-1} q^{[(\frac{r}{2}+\frac{1}{\theta}(n-1)+1)]}$$

for i= 0 and 1, where C and D are two positive magnitudes, independent of n, p, q, and dependent only on θ, ω, and on the coefficients of $F(x)$.

The A and B are provided by special values of the derivatives of polynomials P and Q, given by the first lemma. More precisely, if $\rho Q(x) - P(x) = (\rho - x)^n R(x)$,

$$\begin{aligned} P(x)Q'(x) - P'(x)Q(x) \quad &= (\rho - x)^{n-1} \quad [(\rho - x)(R'(x)Q(x) \\ &\qquad -R(x)Q'(x)) - nR(x)Q(x)]. \end{aligned}$$

Since the left-hand side has rational coefficients, and the irreducible polynomial F is (up to a constant) the minimal polynomial of ρ, the right-hand side is necessarily divisible by $F^{n-1}(x)$. Thus $P(x)Q'(x) - P'(x)Q(x) = F^{n-1}W(x)$, where the degree γ of the polynomial W is less than $\frac{2}{\theta}(n-1)$.

Now it is not possible that the successive μ-th derivatives of $F^{n-1}W(x)$, evaluated at a fraction $\frac{p}{q}$, all vanish for $0 \leq \mu \leq \gamma$, as this would require a factor $(x - \frac{p}{q})^{\gamma+1}$ in $F^{n-1}W(x)$, and thus in $W(x)$. This shows that at least one of the expressions $P^{(a)}(x)Q^{(b)}(x) - P^{(b)}(x)Q^{(a)}(x)$, with $0 \leq a, b \leq \gamma+1$, does not vanish for $x = \frac{p}{q}$.

Thue now defines

$$A = \frac{q^{n+m-\delta}}{\delta!} P^{(\delta)}(\frac{p}{q}), \quad B = \frac{q^{n+m-\delta}}{\delta!} Q^{(\delta)}(\frac{p}{q})$$

and chooses A_0, B_0, resp. A_1, B_1, as the A, B corresponding respectively to $\delta = a$ and $\delta = b$. The choice of a and b shows that the corresponding fractions $\frac{A_0}{B_0}$ and $\frac{A_1}{B_1}$ are distinct.

The proof of the properties of A and B announced in Lemma 2 then comes from the relation

$$\rho Q^{(\delta)} - P^{(\delta)} = \frac{d^\delta}{dx^\delta}[(\rho - x)^n R(x)] = (-1)^n (x - \rho)^{n-\delta}[\cdots]$$

and from the fact that the $\dfrac{R^{(\delta)}(\frac{p}{q}) q^{m-\delta}}{\delta!}$ are bounded.

3.6. Proving Thue's Theorem (sketch). With these two lemmas, Thue proved his main theorem (Theorem I) as follows.

If the theorem were false, one could find (infinitely many) integers p_0, p_1, q_0, q_1, with $q_1 > q_0 > 0$, such that, for $i = 0, 1$

$$q_i \rho - pi = \frac{\epsilon_i}{q_i^h}$$

with $|\epsilon_i| < c$ and $h > \frac{r}{2}$.

For q_0 and $\frac{q_1}{q_0}$ large enough, Thue then shows how to construct θ, ω and n satisfying the conditions of Lemma 2 and such that:

$$\frac{\log q_1 + \log 2}{[(1-\frac{2}{\theta})h - (\frac{r-2}{2}+\frac{1}{\theta})]\log q_0 - \log E} < n-1 < \frac{h\log q_1 - \log 2Cq_0}{(\frac{r}{2}+\frac{1}{\theta})\log q_0 + \log D}$$

for appropriate constants D and E (independent of n, p_0, q_0).

With this choice, and A, B as in Lemma 2, the integer $p_1B - q_1A$ would be strictly smaller than 1, and thus would be 0. This implies that $\frac{p_1}{q_1} = \frac{A}{B}$. But as two different choices of $\frac{A}{B}$ are possible, this cannot be the case. Thue's main theorem is proved.

4. Applications to Diophantine Equations

Thue added three applications (Theorems II, III, IV) to his main theorem. The first one transfers the improvement obtained by Thue in the rational approximation of a real algebraic number to Liouville's result on the growth of its continued-fraction expansion. The second application establishes by Lemma 2 a link between a rational approximation of ρ in $q^{\frac{r}{2}+1+k}$ and one in $q^{\frac{r}{2(k+1)}+1+h}$, for any k and h fixed—the case where $k = \frac{r}{2}-1$, distinguished by Thue, suggesting perhaps that he was aiming at a Roth-like improvement of his theorem. The third application is the most celebrated, in particular because of its future impact on the study of points with integral coordinates on affine algebraic curves; as pointed out, the principle of the method itself was known earlier and used on several occasions by other authors, [35, p. 341].

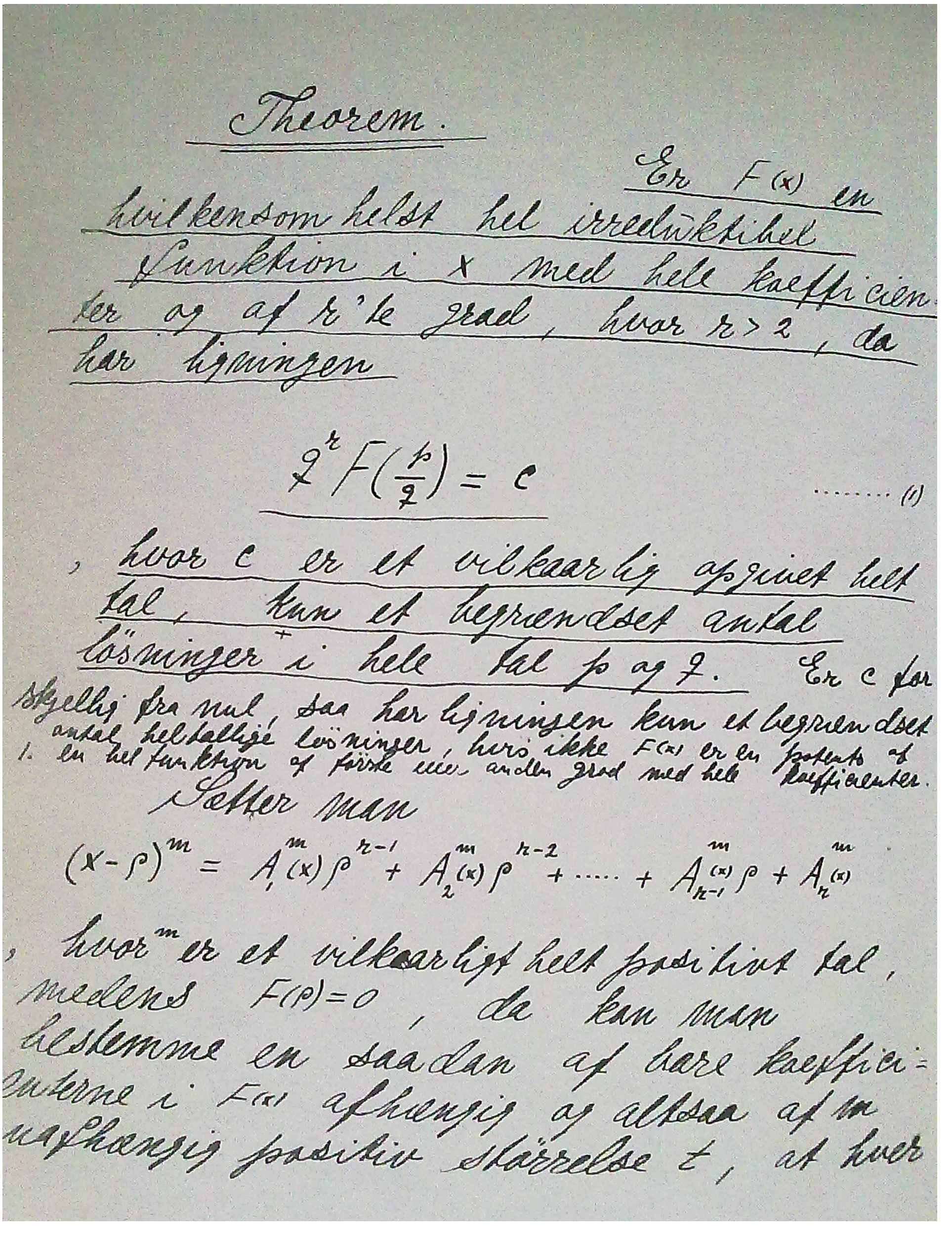
4.1. Integral solutions. This is Thue's Theorem IV in [55]:

Theorem 2. *The equation $U(p,q) = c$, where c is a given constant, and U a homogeneous polynomial with integral coefficients, does not have infinitely many solutions in positive integers p and q, when the degree of U is greater than 2.*

Thue proved in fact that, for any polynomial F of degree r with integral coefficients, the relation $0 < |q^rF(\frac{p}{q})| < cq^h$, for a constant c and a real $h < \frac{r-2}{2}$, has only finitely many solutions.

He uses for this the factorization of $F(x)$, $F(x) = a(x-\rho_1)\cdots(x-\rho_r)$, as in Liouville's proof, which Thue mentions explicitly at this point. Thue then deduces from the assumption $0 < |q^rF(\frac{p}{q})| < cq^h$ that for one of the ρ at least, $|p-q\rho| < c^{\frac{1}{r}}q^{\frac{h}{r}}$. For the other ρ_i, one has $p-\rho_iq = p-\rho q+(\rho-\rho_i)q$, hence $|p-q\rho_i| > -c^{\frac{1}{r}}q^{\frac{h}{r}} + bq > dq$, for a certain d.

Thus $|p-q\rho| < \frac{cq^h}{|a|d^{r-1}q^{r-1}} < \frac{f}{q^{\frac{r}{2}+k}}$, and the main theorem applies.



Theorem.

Er $F(x)$ en hvilkensomhelst hel irreduktibel funktion i x med hele koefficienter og af r'te grad, hvor $r > 2$, da har ligningen

$$q^r F\left(\frac{p}{q}\right) = c \qquad \ldots\ldots\ (1)$$

, hvor c er et vilkaarlig opgivet helt tal, kun et begrændset antal löisninger i hele tal p og q. Er c forskjellig fra nul, saa har ligningen kun et begrændset antal heltallige löisninger, hvis ikke $F(x)$ er en potens af 1. en hel funktion af förste eller anden grad med hele koefficienter.

Setter man

$$(x-\rho)^m = A_1^m(x)\rho^{r-1} + A_2^m(x)\rho^{r-2} + \cdots + A_{r-1}^m(x)\rho + A_r^m(x)$$

, hvor m er et vilkaarligt helt positivt tal, medens $F(\rho)=0$, da kan man bestemme en saadan af bare koefficienterne i $F(x)$ afhængig og altsaa af m uafhængig positiv störrelse t, at hver

FIGURE 3.2. Manuscript of A. Thue's "Om en general i store hele tal uløsbar ligning," published in 1908.

4.2. And yet more results. In the following years, Thue adapted his theorem to a variety of Diophantine applications. He proved for instance, [57, pp. 561–573], that if a, b, c, d, are integers, $a \neq 0$, $b^2 - 4ac \neq 0$, $d \neq 0$, the Diophantine equation

$$ay^2 + by + c = dx^n$$

with n an integer > 3, has only finitely many solutions.

He also extended Theorem IV by replacing the constant c by a polynomial: if $P(x, y)$ and $Q(x, y)$ are two homogeneous polynomials with integral coefficients, of degree p and q, such that $p > 2, p > q$ and P is irreducible, then the equation $P(x, y) = Q(x, y)$ has only finitely many pairs of integral solutions (x, y). Again, if $R(x)$ is a polynomial of degree r, $p > q > r$ and $p < q + r$, the equation $P(x, y) + Q(x, y) + R(x, y) = 0$ has only finitely many solutions (x, y) with x and y relatively prime integers. In addition, he explored a number of equations closely related to Fermat's equation, such as $x^n + (x + k)^n = y^n$, and $ax^n - by^n = k$, with $n > 2$ and k a positive integer.[30] Other applications are described in [44].

5. Coda

As mentioned earlier, Schur hinted at Thue's results after a talk, stressing the difference between the Pell-Fermat equation, $x^2 - dy^2 = 1$, with its infinitely many solutions, and the higher-degree analogues, $x^n - dy^n = 1$, for $n > 2$, which have only finitely many solutions as a consequence of Thue's theorem—the very hint which launched Siegel's interest in the topic, [48, 27]. But this was not an isolated occurence. Despite the difficulties in mathematical communication during World War I, several authors in the 1920s integrated Thue's work into their own: for Diophantine equations, Edmond Maillet, already mentioned, Trygve Nagell, Louis Mordell,[31] among others; Georg Pólya on prime divisors of the values of polynomials at integers, who already in 1917 described Thue's results as "well-known and important,"[37]; Jean Favard in 1929 on the diameter of the set of conjugates of an algebraic numbers, [31], etc.

On the other hand, I have tried to show that Thue inherited not only questions, but also a number of practices from his predecessors and teachers

[30]See in particular [57, pp.358–362, 565–573]. Several of Thue's articles, and the manuscripts left by him, display his ongoing interest in Fermat's Last Theorem. In his obituary of Siegel, Edmund Hlawka even suggested that this interest had triggered his work on rational approximation, [27]. As we have seen above, Thue himself mentioned to Holst a global project including transcendence results.

[31]According to John Cassels, Mordell's research on integral solutions using Thue's theorem led him finally to his celebrated theorem that rational points on curves of genus 1 are finitely generated, see [9]. It was through Mordell and Davenport that Freeman Dyson was informed of Siegel's result and worked out his own 1947 improvement—even, if according to him, his failure to prove the best approximation convinced him to become a physicist instead of a mathematician, see [11, p. 7–8, p. 75-80].

—practices which occur recurrently in his own work. Dirichlet's pigeonhole principle appears not only in Thue's 1909 paper on rational approximation, but also for instance, in the proof of his Remainder Theorem.[32] Polynomials as arithmetical tools are a key to many of Thue's papers, starting with his first communications of the 1890s, and, more specifically, the idea of substituting polynomials for integers, which appears in his 1909 theorem as well as in his later work on Fermat's equation, for instance.

Reconstructing parts of Thue's mathematical environment does not lead of course to diminish his originality or the importance of his work. My purpose here was to give some alternative perspectives to a story which too often reduces the development of number theory at the turn of the twentieth century to that of algebraic number fields, and thus pictures Thue, and in some respects Siegel as well, as completely marginal.[33] Although Thue's personal feeling of isolation at certain times, and his predilection for independent work, are not to be doubted, it does not mean that his mathematics did not engage the interest of his contemporaries.

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[32] See [57, pp. 525–538]. The theorem, used by Thue and others in Diophantine problems, states that if $n > 1$ is an integer, then for any integer a prime to n there exist two positive integers, x and y, less than $E[\sqrt{n}]$, such that $ay \equiv \pm x \bmod n$. This was also proved independently by Alexandre Aubry in 1913.

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