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On the index of degeneracy of a CM abelian variety

par Hiromichi YANAI

Résumé. Nous considérons une variété abélienne dégénérée A de type CM. Alors il existe $m > 0$ tel que l'anneau des cycles de Hodge sur A^m n'est pas engendré par les classes de diviseurs. Nous appelons le plus petit m vérifiant cette propriété *l'indice de dégénérescence* de A.

Dans cet article, nous déterminons l'indice de dégénérescence d'un certain type de variétés abéliennes de type CM. Cela complète un résultat antérieur de H. W. Lenstra, Jr.

Abstract. We consider a degenerate abelian variety A of CM type. Then there exists $m > 0$ such that the ring of Hodge cycles on A^m is not generated by the divisor classes. We call the minimum of such m the *index of degeneracy* of A.

In this paper, we determine the index of degeneracy for a certain type of CM abelian varieties. This supplements a former result of H. W. Lenstra, Jr.

1. Introduction

For an abelian variety A defined over the complex number field $\mathbb{C}$, we call the elements of $H^{2p}(A,\mathbb{Q})\cap H^{p,p}$ the *Hodge cycles* (of codimension p), where $H^{p,q} = H^q(A,\Omega^p)$. It is conjectured that the Hodge cycles are algebraic (this is the Hodge conjecture; for a survey see [1]). Since the divisor classes (the Hodge cycles of codimension 1) are algebraic, we are interested in Hodge cycles that are *not* generated by divisor classes.

We call such cycles *exceptional* (in [9] they are called *sporadic*). Exceptional Hodge cycles cause certain degeneration on various arithmetic objects (*cf.* [10]). Moreover, constructing abelian varieties with exceptional Hodge cycles is related to various combinatorial or group theoretic concepts (*cf.* [4, 5]).

Let A be a CM abelian variety of dimension d. By definition, $\mathrm{End}A \otimes \mathbb{Q}$ contains a CM field K of degree $2d$. Let $S \subset \mathrm{Hom}(K,\mathbb{C})$ be the CM type of A; the representation of K on $H^0(A,\Omega^1)$ is isomorphic to $\bigoplus_{\sigma\in S}\sigma$. We say

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that A is of CM type (K, S). Let $\mathrm{MT}(A)$ be the Mumford-Tate group of A (*cf.* [1]). $\mathrm{MT}(A)$ is an algebraic subtorus of $\mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m$. The dimension of $\mathrm{MT}(A)$ is called the *rank* of A (or the rank of the CM type (K, S)) in certain contexts (*cf.* [7]).

When A is simple, the next two conditions (i) and (ii) are equivalent (*cf.* [1]).

(i) For each positive integer m, the ring of the Hodge cycles on A^m is generated by the divisor classes.

(ii) $\dim \mathrm{MT}(A) = d + 1$.

When A satisfies (one of) these conditions, A is called *stably nondegenerate* and the Hodge conjecture holds for every power of A (*cf.* [2]). If A is not stably nondegenerate (we simply say A is degenerate) then there exists an integer $m > 0$ such that A^m holds an exceptional Hodge cycle. Following F. Hazama [3], we call the minimum of such m the *index of degeneracy* of A.

From this point on, we assume that K is an abelian extension over $\mathbb{Q}$ with the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Then $\dim \mathrm{MT}(A)$ is equal to the number of characters χ of G satisfying $\chi(S) = \sum_{\sigma \in S} \chi(\sigma) \neq 0$ (see [1] for a reference). Note that $\chi(S) = 0$ for every nontrivial even character χ and $\chi_0(S) = d \neq 0$ for the trivial character χ_0. Hence, in this case, the above conditions (i) and (ii) are equivalent to the next (iii).

(iii) For each odd character χ of G, one has $\chi(S) \neq 0$.

In this paper, we prove that if there exists an odd character χ satisfying $\chi(S) = 0$ and the order of χ is a power of 2, then the index of degeneracy of A is equal to 1 (see Theorem 4.1).

Remark. In the CM case, one can provide an example of variety A having index of degeneracy strictly bigger than 1, which was obtained by S. P. White [9].

2. Abelian varieties of Weil type

Let A be a CM abelian variety of type (K, S). We assume that K is an abelian extension over $\mathbb{Q}$ and K contains a proper sub CM field k. H denotes the subgroup of $G = \mathrm{Gal}(K/\mathbb{Q})$ corresponding to k. Put $r = \#H = [K : k]$. For $\tau \in G/H = \mathrm{Gal}(k/\mathbb{Q})$, put

$$H^{1,0}_\tau = \{\omega \in H^{1,0} \mid \forall a \in k, a(\omega) = a^\tau \omega\},$$
$$n_\tau = \dim H^{1,0}_\tau = \#\{\sigma \in S \mid \sigma|_k = \tau\}.$$

If n_τ does not depend on $\tau \in \mathrm{Gal}(k/\mathbb{Q})$, we say that A is of *Weil type*. It is easy to see that the condition is equivalent to $n_\tau = n_{\tau\rho} = \frac{r}{2}$ for each τ, where ρ denotes the complex conjugation. For such A, $\chi(S) = 0$ for the odd characters χ of G which are trivial on H. If this is the case, A is degenerate and there exists an exceptional Hodge cycle of codimension $\frac{r}{2}$ (*cf.* [6]). In particular, the index of degeneracy of A is equal to 1.

3. Lenstra's result

When a CM abelian variety A is degenerate, we want to determine the index of degeneracy of A. H. W. Lenstra, Jr. gives an answer for some cases. Here we recall his argument briefly. For more information, see [9]. (The expressions in [9] are slightly different from ours.)

Let A be a simple CM abelian variety of type (K, S). Let us assume that K is an abelian extension over $\mathbb{Q}$ and that there exists an odd character χ of $G = \mathrm{Gal}(K/\mathbb{Q})$ with $\chi(S) = 0$; hence A is degenerate.

When χ is faithful, G is cyclic. We denote a generator of G by γ and the order of γ by $2t$. Put

$$\boldsymbol{h} = \prod_{p|t} (\epsilon + \rho\gamma^{\frac{2t}{p}}),$$

where p runs over the *odd* primes dividing t and ϵ is the unit element of G.

The element $\boldsymbol{h}$ lies in the group ring $\mathbb{Z}[G]$. We can see that the coefficients in $\boldsymbol{h}$ are 1 or 0, hence $\boldsymbol{h}$ is naturally regarded as a subset of G. For each $\sigma \in G$, we take a nonzero $\omega_\sigma \in H^1(A, \mathbb{C})$ such that $a(\omega_\sigma) = a^\sigma \omega_\sigma$ for each $a \in K$. Such ω_σ is unique up to a constant multiple; we have $H^1(A, \mathbb{C}) = \bigoplus_{\sigma \in G} \mathbb{C}\omega_\sigma$.

Then the element

$$\bigwedge_{\sigma \in \boldsymbol{h}} \omega_\sigma$$

(one implicitly fixes an order for taking this wedge product) is an exceptional Hodge cycle on A. More precisely, it is an element of $(H^{2q}(A, \mathbb{Q}) \cap H^{q,q}) \otimes \mathbb{C}$ for some q and is not generated by divisor classes. This implies that the index of degeneracy of A is equal to 1.

When χ is not faithful, we take the above $\boldsymbol{h}$ for $G/\mathrm{Ker}\chi$, then its pullback to G works.

In the definition of $\boldsymbol{h}$, we have used the odd prime factors of t. When t is a power of 2, such a prime number does not exist and the above argument doesn't work.

4. The 2-power case

In this section, we consider the case where t is a power of 2. In this case, the index of degeneracy of A is also 1, but the reason is different from that of Lenstra's case.

Theorem 4.1. *Let A be a CM abelian variety of dimension d as in Section 3. Assume that there exists an odd character χ of $G = Gal(K/\mathbb{Q})$ with $\chi(S) = 0$ and the order of χ $(= 2t)$ is a power of 2. Then A is of Weil type; the index of degeneracy of A is equal to 1.*

Proof. Put $t = 2^{s-1}$ with $s > 0$. Let H be the kernel of χ and k be the subfield of K corresponding to H. The quotient $G/H = \mathrm{Gal}(k/\mathbb{Q})$ is a cyclic group of order $2t$. Let us denote by γ (a representative of) a generator of G/H. Then $\zeta = \chi(\gamma)$ is a primitive 2^s-th root of unity. Let $G = \bigcup_{i=0}^{2t-1} H\gamma^i$ be the coset decomposition of G by H, where $H\gamma^t$ is the coset of the complex conjugation ρ. For each i, put $n_i = \#(S \cap H\gamma^i) = \dim H_\tau^{1,0}$, where $\tau \in \mathrm{Gal}(k/\mathbb{Q})$ is corresponding to $H\gamma^i$. Then $n_i + n_{t+i} = \#H = \frac{d}{t}$ $(0 \le i \le t-1)$.

We have

$$0 = \sum_{\sigma \in S} \chi(\sigma) = \sum_{i=0}^{2t-1} n_i \chi(\gamma^i) = \sum_{i=0}^{t-1} (n_i - n_{t+i}) \zeta^i.$$

Since ζ is a primitive 2^s-th root of unity, its degree over $\mathbb{Q}$ is $2^{s-1} = t$. This implies $n_i = n_{t+i}$ $(0 \le i \le t-1)$. If H is trivial (*i.e.* $k = K$), then $n_i = 0$ or 1 for each i, but this is impossible because $n_i + n_{t+i} = 1$. So k is a proper CM subfield of K and A is of Weil type with respect to k; the index of degeneracy of A is equal to 1. □

5. Example: CM Jacobians

In this section, we give examples of abelian varieties satisfying the condition in the previous section. These are CM Jacobian varieties treated in [8]. For these abelian varieties, we can determine the dimensions of the Mumford-Tate groups.

Let $p > 5$ be a prime number and ζ_p be a primitive p-th root of unity. We denote the minimal polynomial of $-(\zeta_p + \zeta_p^{-1})$ by $g(x) \in \mathbb{Z}[x]$. Then the Jacobian variety A of the algebraic curve $y^2 = x \cdot g(x^2 - 2)$ is a simple CM abelian variety of dimension $\frac{p-1}{2}$. Put $K = \mathbb{Q}(\sqrt{-1}, \zeta_p + \zeta_p^{-1})$ then $G = \mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \left((\mathbb{Z}/p\mathbb{Z})^\times / \{\pm 1\} \right)$ and $\mathrm{End}A \otimes \mathbb{Q} \cong K$.

According to [8], the CM type S of A is:

$$\text{when } p \equiv 1 \pmod 4, \quad S = \{(0,1), (1,2), (0,3), \dots, (1, \frac{p-1}{2})\},$$

$$\text{when } p \equiv 3 \pmod 4, \quad S = \{(0,1), (1,2), (0,3), \dots, (0, \frac{p-1}{2})\}.$$

Let ψ be the nontrivial character of $(\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z}$ and ξ_0 be the trivial character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Then the product $\psi \times \xi_0$ can be viewed as an odd character of G; its order is 2.

When $p \equiv 1 \pmod 4$, one has $(\psi \times \xi_0)(S) = 1 - 1 + 1 - \cdots - 1 = 0$. Hence by Theorem 4.1, the abelian variety A is of Weil type with $k = \mathbb{Q}(\sqrt{-1})$ and the index of degeneracy of A is equal to 1.

Moreover, representing an odd character χ of G by $\chi = \psi \times \xi$ (where ξ is an even character mod p), we can describe the value $\chi(S)$ by the generalized Bernoulli numbers $B_{1,\chi} = \frac{1}{4p} \sum_{a=1}^{4p} \chi(a)a$. Here we are regarding χ as a character mod $4p$.

In fact, for $\xi \neq \xi_0$, we can deduce:

when $p \equiv 1 \pmod 4$,

$$\chi(S) = \xi(1) - \xi(2) + \xi(3) - \cdots - \xi(\frac{p-1}{2}) = \bar{\xi}(2) B_{1,\chi},$$

when $p \equiv 3 \pmod 4$,

$$\chi(S) = \xi(1) - \xi(2) + \xi(3) - \cdots + \xi(\frac{p-1}{2}) = -\bar{\xi}(2) B_{1,\chi}.$$

We know $B_{1,\chi} \neq 0$. When $p \equiv 3 \pmod 4$, one has $(\psi \times \xi_0)(S) \neq 0$. Hence the dimension of the Mumford-Tate group $\mathrm{MT}(A)$ of A is:

$$\text{when } p \equiv 1 \pmod 4,\ \dim \mathrm{MT}(A) = \dim A,$$
$$\text{when } p \equiv 3 \pmod 4,\ \dim \mathrm{MT}(A) = \dim A + 1.$$

In particular, when $p \equiv 3 \pmod 4$, A is stably nondegenerate; the Hodge conjecture holds for every power of A.

Remark. When $p \equiv 1 \pmod 4$ and when $\xi(*) = \left(\frac{*}{p}\right)$ (the quadratic residue symbol), the above calculations imply the following (probably well known) equalities concerning the class number h_p of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$:

$$\left(\frac{1}{p}\right) - \left(\frac{2}{p}\right) + \left(\frac{3}{p}\right) - \cdots - \left(\frac{\frac{p-1}{2}}{p}\right) = -\left(\frac{2}{p}\right) h_p,$$

$$\left(\frac{1}{p}\right) - \left(\frac{2}{p}\right) - \left(\frac{3}{p}\right) + \left(\frac{4}{p}\right) + \left(\frac{5}{p}\right) - \cdots + (-1)^{\frac{p-1}{4}} \left(\frac{\frac{p-1}{2}}{p}\right) = h_p.$$

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